

C'est pour toi que je joue, Alf c'est pour toi,  
 Tous les autres m'écoutent, mais toi tu m'entends ...  
 Exilé d'Amsterdam vivant en Australie,  
 Ulysse qui jamais ne revient sur ses pas ...  
 Je suis de ton pays, métèque comme toi,  
 Quand il faudra mourir, on se retrouvera<sup>1</sup>

*To the memory of Alf van der Poorten*

# ON THE VANISHING OF IWASAWA'S CONSTANT $\mu$ FOR THE CYCLOTOMIC $\mathbb{Z}_p$ -EXTENSIONS OF $CM$ NUMBER FIELDS.

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ABSTRACT. We prove that  $\mu = 0$  for the cyclotomic  $\mathbb{Z}_p$ -extensions of  $CM$  number fields.

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## 1. INTRODUCTION

Iwasawa gave in his seminal paper [4] from 1973 examples of  $\mathbb{Z}_p$ -extensions in which the structural constant  $\mu \neq 0$ . In the same paper, he proved that

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<sup>1</sup>Free after Georges Moustaki, “Grand-père”

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if  $\mu = 0$  for the cyclotomic  $\mathbb{Z}_p$ -extension of some number field  $\mathbb{K}$ , then the constant vanishes for any cyclic  $p$ -extension of  $\mathbb{K}$  – and thus for any number field in the pro- $p$  solvable extension of  $\mathbb{K}$ . Iwasawa also suggested in that paper that  $\mu$  should vanish for the cyclotomic  $\mathbb{Z}_p$ -extension of all number fields, a fact which is sometimes called *Iwasawa's conjecture*. The conjecture has been proved by Ferrero and Washington [3] for the case of abelian fields. In this paper, we give an independent proof, which holds for all CM fields:

**Theorem 1.1.** *Let  $\mathbb{K}$  be a CM number field and  $p$  an odd prime. Then Iwasawa's constant  $\mu$  vanishes for the cyclotomic  $\mathbb{Z}_p$ -extension  $\mathbb{K}_\infty/\mathbb{K}$ .*

**1.1. Notations and basic facts on decomposition of  $\Lambda$ -modules.** In this paper  $p$  is an odd prime. In the sequel, the Iwasawa constant  $\mu$  for some number field  $\mathbb{K}$  will always refer to the  $\mu$ -invariant for the  $\mathbb{Z}_p$ -cyclotomic extension of  $\mathbb{K}$ . We shall denote number fields by black board bold characters, e.g.  $\mathbb{K}, \mathbb{L}$ , etc. If  $\mathbb{K}$  is a number field, its cyclotomic  $\mathbb{Z}_p$ -extension is  $\mathbb{K}_\infty$  and  $\mathbb{B}_\infty/\mathbb{Q}$  is the  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$ , so  $\mathbb{K}_\infty = \mathbb{K} \cdot \mathbb{B}_\infty$ . We denote as usual by  $\Gamma$  the Galois group  $\text{Gal}(\mathbb{K}_\infty/\mathbb{K})$  and let  $\tau \in \Gamma$  be a topological generator.

Let  $\mathbb{B}_1 = \mathbb{Q}$  and  $\mathbb{B}_n \subset \mathbb{B}_\infty$  be the intermediate extensions of  $\mathbb{B}_\infty$  and let the counting be given by  $[\mathbb{B}_n : \mathbb{Q}] = p^{n-1}$ , so  $\mathbb{B}_n = \mathbb{B}_\infty \cap \mathbb{Q}[\zeta_{p^n}]$ . Let  $\kappa > 0$  be the integer for which  $\mathbb{K} \cap \mathbb{B}_\infty = \mathbb{B}_\kappa$ ; then  $\mu_{p^\kappa} \subset \mathbb{K}$  and  $\mu_{p^{\kappa+1}} \not\subset \mathbb{K}$ . We shall use a counting of the intermediate fields of  $\mathbb{K}_\infty$  that reflects this situation and let  $\mathbb{K}_0 = \mathbb{K}_1 = \dots = \mathbb{K}_\kappa = \mathbb{K}$  and  $[\mathbb{K}_{\kappa+1} : \mathbb{K}] = p$ , etc. The constant  $\kappa$  can be determined in the same way for any number fields and we adopt the same counting in any cyclotomic  $\mathbb{Z}_p$ -extension. This way, for  $n \geq \kappa$  we always have  $\mu_{p^n} \subset \mathbb{K}_n$ . We let  $\gamma \in \text{Gal}(\mathbb{B}_\infty/\mathbb{Q})$  be a topological generator of  $\text{Gal}(\mathbb{B}_\infty/\mathbb{Q})$  and let  $\tau \in \Gamma := \text{Gal}(\mathbb{K}_\infty/\mathbb{K})$  be a topological generator for  $\Gamma$ . We may thus assume that  $\tau = \gamma^{p^{\kappa-1}}$  for some lift  $\gamma \in \text{Gal}(\mathbb{K}_\infty/\mathbb{Q})$  of  $\gamma$  and write as usual  $T = \tau - 1$ ,  $\Lambda = \mathbb{Z}_p[[T]]$  and

$$\begin{aligned} \omega_n &= \tau^{p^{n-\kappa}} - 1 = (T + 1)^{p^{n-\kappa}} - 1, \\ \nu_{m,n} &= \omega_m / \omega_n, \quad \text{for } m > n \geq \kappa. \end{aligned}$$

Since  $\mathbb{K} = \mathbb{K}_1 = \mathbb{K}_\kappa$ , we may also write  $\nu_{m,1} = \nu_{m,\kappa}$ . Note that the special numeration of fields which we introduce in order to ascertain that  $\mu_{p^n} \subset \mathbb{K}_n$  induces a shift in the exponents in the definition of  $\omega_n$ . In terms of  $\gamma$  we recover the classical definitions, but our exposition is made in terms of the topological generator  $\tau$ . Note that the base field  $\mathbb{K}$  with respect to which we shall bring our proof, is still to be defined, and will be a CM on in which we assume that  $\mu > 0$ , and in which some useful additional conditions are fulfilled.

The  $p$ -Sylow subgroups of the class groups  $\mathcal{C}(\mathbb{K}_n)$  of the intermediate fields of  $\mathbb{K}_n$  are denoted as usual by  $A_n = A_n(\mathbb{K}) = (\mathcal{C}(\mathbb{K}_n))_p$ ; the explicit reference to the base field  $\mathbb{K}$  will be used when we refer simultaneously to sequences of class groups related to different base fields. Traditionally,  $A = A(\mathbb{K}) = \varprojlim A_n$ . We use the projective limit, which we denote by

${}^{\text{p}}\mathbf{A}^-(\mathbb{K}) = \varprojlim (A_n)$ , as explained in detail below. The maximal  $p$ -abelian unramified extensions of  $\mathbb{K}_n$  are denoted by  $\mathbb{H}_n = \mathbb{H}_n(\mathbb{K}) \supset \mathbb{K}_n$  and  $X_n = \text{Gal}(\mathbb{H}_n/\mathbb{K}_n)$ . The projective limit with respect to restriction maps is  $X = \varprojlim (X_n)$ : it is a Noetherian  $\Lambda$ -torsion module on which complex conjugation acts inducing the decomposition  $X = X^+ \oplus X^-$ . The Artin maps  $\varphi : A_n \rightarrow X_n$  are isomorphisms of finite abelian  $p$ -groups; whenever the abelian extension  $\mathbb{M}/\mathbb{K}$  is clear in the context, we write  $\varphi(a)$  for the Artin Symbol  $\left(\frac{\mathbb{M}/\mathbb{K}}{a}\right)$ , where  $a \in \mathcal{C}(\mathbb{K})$  if  $\mathbb{M}$  is unramified, or  $a$  is an ideal of  $\mathbb{K}$  otherwise. Complex conjugation acts on class groups and Galois groups, inducing direct sum decompositions in plus and minus parts:

$$A_n = A_n^+ \oplus A_n^-, \quad X_n = X_n^+ \oplus X_n^-, \quad \text{etc.}$$

The idempotents generating plus and minus parts are  $\frac{1 \pm j}{2}$ ; since 2 is a unit in the ring  $\mathbb{Z}_p$  acting on these groups, we also have  $Y^+ = (1 + j)Y$  and  $Y^- = (1 - j)Y$ , for  $Y \in \{A_n, X_n, X, \dots\}$ . Throughout the paper, we shall use, by a slight abuse of notation and unless explicitly specified otherwise, the additive writing for the group ring actions. This is preferable for typographical reasons. If  $M$  is some Noetherian  $\Lambda$ -torsion module on which  $j$  acts, inducing a decomposition  $M = M^+ \oplus M^-$ , we write  $\mu^-(M) = \mu(M^-)$ ,  $\lambda^-(M) = \lambda(M^-)$ , etc, for the Iwasawa constants of this module. In the case when  $M = {}^{\text{p}}\mathbf{A}^-(\mathbb{M})$  or  $M = X(\mathbb{M})$  is attached to some number field  $\mathbb{M}$ , we simply write  $\mu^-(\mathbb{M})$  or  $\mu({}^{\text{p}}\mathbf{A}^-(\mathbb{M}))$ .

By assumption on  $\mathbb{K}$ , the norms  $\mathbf{N}_{\mathbb{K}_m/\mathbb{K}_n} : A_m \rightarrow A_n$  are surjective for all  $m > n > 0$ . Therefore, the sequence  $(A_n)_{n \in \mathbb{N}}$  is projective with respect to the norm maps and we denote their projective limit by  ${}^{\text{p}}\mathbf{A} = \varprojlim_n A_n$ . The Artin map induces an isomorphism of compact  $\Lambda$ -modules  $\varphi : {}^{\text{p}}\mathbf{A} \rightarrow X$ . The elements of  ${}^{\text{p}}\mathbf{A}$  are norm coherent sequences  $a = (a_n)_{n \in \mathbb{N}} \in {}^{\text{p}}\mathbf{A}$  with  $a_n \in A_n$  for  $n \geq \kappa$ ; we let  $a_0 = a_1 = \dots = a_\kappa$ . It is customary to identify  $X$  with  ${}^{\text{p}}\mathbf{A}$  via the Artin map; we shall not use injective limits here, but make explicit reference to the projective limit  ${}^{\text{p}}\mathbf{A}$ .

It is a folklore fact that if  $\mu$  vanishes for the cyclotomic  $\mathbb{Z}_p$ -extension of some CM field  $\mathbb{K}_{\text{start}}$ , then it vanishes for any finite algebraic extension thereof. We prove this in Fact 3.1 in the Appendix. In order to prove the Theorem 1.1 we shall need to tailor some base field  $\mathbb{K}/\mathbb{K}_{\text{start}}$ , which is a Galois CM extension of  $\mathbb{Q}$  and enjoys some additional conditions that shall be discussed below; of course, we assume that  $\mu(\mathbb{K}) > 0$ . Before describing the construction of  $\mathbb{K}$  however, we need to introduce some definitions and auxiliary constructions.

**1.2. Decomposition and Thaine Shifts.** Let  $M$  be a Noetherian  $\Lambda$ -torsion module. It is associated to an *elementary* Noetherian  $\Lambda$ -torsion module  $\mathcal{E}(M) \sim M$  defined by:

$$\begin{aligned} \mathcal{E}(M) &= \mathcal{E}_\lambda(M) \oplus \mathcal{E}_\mu(M), & \text{with} \\ \mathcal{E}_\mu(M) &= \bigoplus_{i=1}^r \Lambda/(p^{e_i} \Lambda), & \mathcal{E}_\lambda(M) &= \bigoplus_{j=1}^{r'} \Lambda/(f_j^{e'_j} \Lambda), \end{aligned}$$

where all  $e_i, e'_j > 0$  and  $f_j \in \mathbb{Z}_p[T]$  are irreducible distinguished polynomials. The pseudoisomorphism  $M \sim \mathcal{E}(M)$  is given by the exact sequence

$$(1) \quad 1 \rightarrow K_1 \rightarrow M \rightarrow \mathcal{E}(M) \rightarrow K_2 \rightarrow 1,$$

in which the kernel and cokernel  $K_1, K_2$  are finite. We define  $\lambda$ - and  $\mu$ -parts of  $M$  as follows:

**Definition 1.2.** *Let  $M$  be a Noetherian  $\Lambda$ -torsion module. The  $\lambda$ -part  $\mathcal{L}(M)$  is the maximal  $\Lambda$ -submodule of  $M$  of finite  $p$ -rank. The  $\mu$ -part  $\mathcal{M}(M)$  is the  $\mathbb{Z}_p$ -torsion submodule of  $M$ ; it follows from the Weierstraß Preparation Theorem that there is some  $m > 0$  such that  $\mathcal{M}(M) = M[p^m]$ . The maximal finite  $\Lambda$ -submodule of  $M$  is  $\mathcal{F}(M)$ , its finite part. By definition,  $\mathcal{L}(M) \cap \mathcal{M}(M) = \mathcal{F}(M)$ .*

*Let the module  $\mathcal{D}(M) = \mathcal{L}(M) + \mathcal{M}(M)$  be the decomposed submodule of  $M$ . Then for all  $x \in \mathcal{D}(M)$  there are  $x_\lambda \in \mathcal{L}(M), x_\mu \in \mathcal{M}(M)$  such that  $x = x_\lambda + x_\mu$ , the decomposition being unique iff  $\mathcal{F}(M) = 0$ . The pseudoisomorphism  $M \sim \mathcal{E}(M)$  implies that  $[M : \mathcal{D}(M)] < \infty$ .*

*If  $x \in M \setminus \mathcal{D}(M)$ , the  $L$ - and the  $D$ -orders of  $x$  are, respectively*

$$(2) \quad \begin{aligned} \ell(x) &= \min\{j > 0 : p^j x \in \mathcal{L}(M)\}, \quad \text{and} \\ \delta(x) &= \min\{k > 0 : p^k x \in \mathcal{D}(M)\} \leq \ell(x). \end{aligned}$$

*We say that a Noetherian  $\Lambda$ -module  $M$  of  $\mu$ -type is rigid if the map  $\psi : M \rightarrow \mathcal{E}(M)$  is injective. Rigid modules have the fundamental property that for any distinguished polynomial  $g(T) \in \mathbb{Z}_p[T]$  and any  $x \in M$*

$$(3) \quad g(T)x = 0 \quad \Leftrightarrow \quad x = 0.$$

Note that for CM base fields  $\mathbb{M}$ , the modules  $\mathcal{M}({}^pA^-(\mathbb{M}_\infty))$ , where  $\mathbb{M}_\infty$  is the cyclotomic  $\mathbb{Z}_p$ -extension, are rigid. The following fact about decomposition is proved in the last section of the Appendix.

**Proposition 1.3.** *Let  $\mathbb{M}$  be a number field, let  $\mathbb{T}_n \supset \mathbb{H}(\mathbb{M}_n) \supset \mathbb{M}_n$  be the ray class fields to some fixed ray  $\mathfrak{M}_0 \subset \mathcal{O}(\mathbb{M})$  and  $M = {}^pA^-(\mathbb{M})$  be the projective limit of the galois groups  $T_n = \text{Gal}(\mathbb{T}_n/\mathbb{M}_n)$ . Assume in addition that the following condition is satisfied by  $\mathbb{M}$ : if  $r = p\text{-rk}(\mathcal{L}(M))$  and  $L_1 = \mathbf{N}_{\mathbb{M}_\infty/\mathbb{M}_1}(\mathcal{L}(M))$  then*

$$(4) \quad p\text{-rk}(L_1) = r \quad \text{and} \quad \text{ord}(x) > p^2 \text{ for all } x \in L_1 \setminus L_1^p.$$

*If these hypotheses hold and  $x \in M$  is such that  $px \in \mathcal{D}(M)$ , then  $T^2x \in \mathcal{D}(M)$ .*

We observe that the condition (4) can be easily satisfied by eventually replacing an initial field  $\mathbb{M}$  with some extension, as explained in Fact 3.2 of Appendix 3.1

Next we define Thaine shifts and lifts. Let  $\mathbb{M}$  be a CM Galois field and consider  $a = (a_n)_{n \in \mathbb{N}} \in {}^pA^-(\mathbb{M})$ , some norm coherent sequence. We assume that the norm maps  $\mathbf{N}_{\mathbb{M}_n/\mathbb{M}_{n'}} : A_n^-(\mathbb{M}) \rightarrow A_{n'}^-(\mathbb{M})$  are surjective for all  $n > n' \geq 1$  and fix some integer  $m > \kappa(\mathbb{M})$  – with  $\mathbb{M} \cap \mathbb{B}_\infty = \mathbb{B}_{\kappa(\mathbb{M})}$  – and

a totally split prime  $\mathfrak{q} \in a_m$ , which is inert in  $\mathbb{M}_\infty/\mathbb{M}_m$ . Let  $q \in \mathbb{Q}$  be the rational prime below  $\mathfrak{q}$  and assume that  $q \equiv 1 \pmod{p}$ . Let  $\mathbb{F} \subset \mathbb{Q}[\zeta_q]$  be the subfield of degree  $p$  in the  $q$ -th cyclotomic extension. We let  $\mathbb{L}_n = \mathbb{M}_n \cdot \mathbb{F}$  and  $\mathbb{L}_\infty = \mathbb{M}_\infty \cdot \mathbb{F}$ . The tower  $\mathbb{L}_\infty/\mathbb{L}$  is the *inert Thaine shift* of the initial cyclotomic extension  $\mathbb{M}_\infty/\mathbb{M}$ , induced by  $\mathfrak{q} \in a_m$ . Let  $\mathfrak{Q} \subset \mathbb{L}_m$  be the ramified prime above  $\mathfrak{q}$ . According to Lemma 3.5 in the Appendix, we may apply Tchebotarew's Theorem in order to construct a sequence  $b = (b_n)_{n \in \mathbb{N}} \in {}^p\mathcal{A}^-(\mathbb{L})$  such that  $b_m = [\mathfrak{Q}]$  is the class of  $\mathfrak{Q}$  and  $\mathbf{N}_{\mathbb{L}_n/\mathbb{M}_n}(b_n) = a_n$  for all  $n \in \mathbb{N}$ . In the projective limit, we then also have  $\mathbf{N}_{\mathbb{L}_\infty/\mathbb{M}_\infty}(b) = a$ . A sequence determined in this way will be denoted a *Thaine lift* of  $a$ . It is not unique.

Let  $F = \text{Gal}(\mathbb{F}/\mathbb{Q})$  be generated by  $\nu \in F$ ; we write  $s := \nu - 1$  and  $\Phi_p(\nu) = \frac{(s+1)^p - 1}{s}$ . By using the identity  $\frac{x^p - 1}{x - 1} = \frac{(y+1)^p - 1}{y} = y^{p-1} + O(p)$ , we see that the algebraic norm verifies

$$(5) \quad \mathcal{N} := \sum_{i=0}^{p-1} \nu^i = \Phi_p(\nu) = pu(s) + s^{p-1} = p + sf(s),$$

$$f \in \mathbb{Z}_p[X] \setminus p\mathbb{Z}_p[s], \quad \in (\mathbb{Z}/(p^N \cdot \mathbb{Z})[s])^\times, \forall N > 0.$$

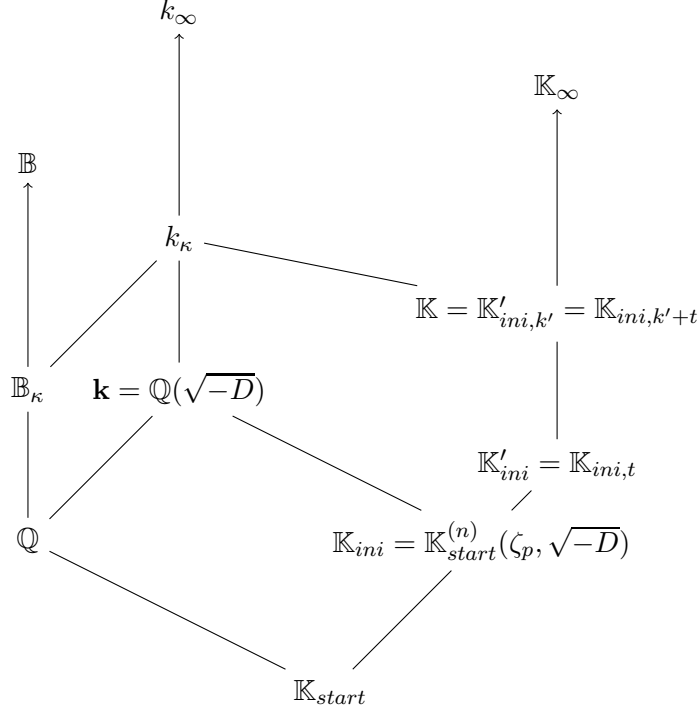
Since  $\mathfrak{q}$  is totally split in  $\mathbb{M}_m/\mathbb{Q}$ , the extensions  $\mathbb{L}_n/\mathbb{Q}$  are Galois and  $F$  commutes with  $\text{Gal}(\mathbb{M}_m/\mathbb{Q})$ , for every  $m$  and  $\text{Gal}(\mathbb{L}_n/\mathbb{M}_n) \cong F$ .

**1.3. Constructing the base field.** For our proof we shall choose a base field  $\mathbb{K}$  as follows. Start with some CM field  $\mathbb{K}_{\text{start}}$  for which one assumes that  $\mu > 0$ , and let  $-D$  be a quadratic non-residue modulo  $p$  – so  $\left(\frac{-D}{p}\right) = -1$  – and  $\mathbf{k} = \mathbb{Q}[\sqrt{-D}]$  be a quadratic imaginary extension. Our start field should be galois and contain the  $p$ -th roots of unity. We require thus that  $\mathbb{K} \supset \mathbb{K}_{\text{start}}^{(n)}[\zeta_p, \sqrt{-D}]$ .

We additionally expect that the primes that ramify in  $\mathbb{K}_\infty/\mathbb{K}$  be totally ramified and the norms  $N_{n,m} : A(\mathbb{K}_n) \rightarrow A(\mathbb{K}_m)$  be surjective, so  $\mathbb{K}_\infty \cap \mathbb{H}(\mathbb{K}) = \mathbb{K}$ . We also require that the exponent  $\exp(\mathcal{M}({}^p\mathcal{A}^-(\mathbb{K}))) \geq p^2$ , which can be achieved by means of Fact 3.2. The first step of the construction consists thus in replacing  $\mathbb{K}_{\text{start}}$  by an initial field  $\mathbb{K}_{\text{ini}} = \mathbb{K}_{\text{start}}^{(n)}[\zeta_p, \sqrt{-D}]$ . If  $\exp(\mathcal{M}({}^p\mathcal{A}^-(\mathbb{K}_{\text{ini}}))) = p$ , then replace  $\mathbb{K}_{\text{ini}}$  by some Thaine shift thereof, in order to increase the exponent. We need to fulfill the condition (4) required in Proposition 1.3; for this we determine an integer  $t$  as follows: let  $L = \mathcal{L}({}^p\mathcal{A}^-(\mathbb{K}_{\text{ini}}))$  and  $r = p\text{-rk}(L) = \lambda(\mathbb{K}_{\text{ini}})$ . Let  $L_t = \mathbf{N}_{\mathbb{K}_{\text{ini},\infty}/\mathbb{K}_{\text{ini};t}}(L)$ . We then require that

$$(6) \quad p\text{-rk}(L_t) = r \quad \text{and} \quad \text{ord}(x) > p^2 \quad \text{for all } x \in L_t \setminus L_t^p.$$

With this, we let  $\mathbb{K}'_{\text{ini}} = \mathbb{K}_{\text{ini},t} \subset \mathbb{K}_{\text{ini},\infty}$ . Finally we apply the Proposition 1.3 in order to choose a further extension of  $\mathbb{K}'_{\text{ini}}$  in the same cyclotomic  $\mathbb{Z}_p$ -extension, that yields some simple decomposition properties for Thaine lifts.

FIGURE 1. Construction of the base-field  $\mathbb{K}$ 

Let  $\kappa' = \kappa(\mathbb{K}'_{ini})$  and  $\tilde{\tau} = \gamma^{p^{\kappa'-1}}$  generate  $\tilde{\Gamma} := \text{Gal} \mathbb{K}'_{ini,\infty} / \mathbb{K}'_{ini}$ . Recall that  $\gamma$  is a generator of  $\text{Gal}(\mathbb{B}_\infty / \mathbb{B})$ , which explains the definition of  $\tilde{\tau}$ ; finally,  $\tilde{T} = \tilde{\tau} - 1$  and we let  $p^b$  be the exponent of  $\mathcal{M}({}^pA^-(\mathbb{K}'_{ini}))$ . We let  $k'$  be such that  $\tilde{\omega}_{k'} \in (p^{b+1}, T^{2(b+1)})$  and define  $\mathbb{K} = \mathbb{K}'_{ini,k'} = \mathbb{K}_{ini,k'+t}$  and let finally  $\kappa = \kappa(\mathbb{K})$  and  $\tau = \gamma^{p^{\kappa-1}}$ ,  $T = \tau - 1$ , etc. We conclude from Proposition 1.3 and the choice of  $k'$  that

$$(7) \quad T \cdot {}^pA^-(\mathbb{K}) \subset \mathcal{D}({}^pA^-(\mathbb{K})).$$

Moreover:

**Remark 1.4.** Suppose that  $\mathbb{L}/\mathbb{K}$  is a Thaine shift and  $y \in {}^pA^-(\mathbb{L})$  is such that either

1.  $py = x + w$  with  $x \in \iota_{\mathbb{L}/\mathbb{K}}({}^pA^-(\mathbb{K}))$  and  $w \in \mathcal{M}({}^pA^-(\mathbb{L}))$ , or
2.  $p^{b+1}y \in \mathcal{L}({}^pA^-(\mathbb{L}))$ .

Then  $Ty \in \mathcal{D}({}^pA^-(\mathbb{L}))$  too.

The second point is a direct consequence of the choice of  $k'$  and of Proposition 1.3. For the first, since  $x \in {}^pA^-(\mathbb{K})$ , we know that  $p^bx \in \mathcal{L}({}^pA^-(\mathbb{K}))$ , so  $p^{b+1}y = p^bx - p^bw \in \mathcal{L}({}^pA^-(\mathbb{K})) + \mathcal{M}({}^pA^-(\mathbb{L})) \subset \mathcal{D}({}^pA^-(\mathbb{L}))$ , and the fact follows from point 2.

The construction of the Thaine shift is shown in the Figure 2

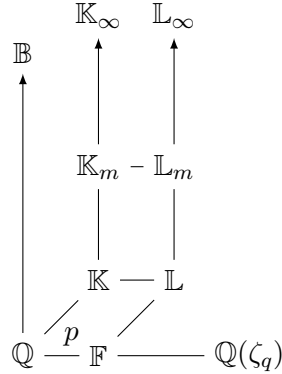


FIGURE 2. Thaine shift and lift

This concludes the sequence of steps for the construction of the base field  $\mathbb{K}$ , which are reflected in the figure 1. We review the conditions fulfilled by this field:

0. The field  $\mathbb{K}' = \mathbb{K}_{start}^{(n)}[\zeta_p, \sqrt{-D}]$  and  $\mathbb{K}_{ini} = \mathbb{K}'_s$  with  $t$  subject to (6).
1. The field  $\mathbb{K}$  is a Galois CM extension  $\mathbb{K}/\mathbb{Q}$  which contains the  $p$ -th roots of unity and such that  $\mu(\mathbb{K}) > 0$  for the cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{K}$ . The primes that ramify in  $\mathbb{K}_\infty/\mathbb{K}$  are totally ramified.
2. We have  $T \cdot (\text{PA}^-(\mathbb{K})) \subset \mathcal{D}(\text{PA}^-(\mathbb{K}))$  and the properties in Remark 1.4 are verified.
3. The numbering of intermediate fields starts from  $\kappa$ , where  $\mathbb{K} \cap \mathbb{B} = \mathbb{B}_\kappa$ .
4. The exponent  $\exp(\mathcal{M}(\text{PA}^-(\mathbb{K}))) \geq p^2$ .
5. The field  $\mathbb{K}$  contains an imaginary quadratic extension  $\mathbf{k} = \mathbb{Q}[\sqrt{-d}] \subset \mathbb{K}$  which has trivial  $p$ -part of the class group.

**1.4. Plan of the paper.** We choose a base field  $\mathbb{K}$  as shown in the previous section and a norm coherent sequence

$$a = (a_n)_{n \in \mathbb{N}} \in \mathcal{M}(\text{PA}^-(\mathbb{K})) \setminus (p \cdot \text{PA}^-(\mathbb{K}) + (p, T)\mathcal{M}(\text{PA}^-(\mathbb{K}))).$$

We note that condition 0. in the choice of  $\mathbb{K}_{ini}$  readily implies that  $\text{ord}(a) = \text{ord}(a_1)$ , so  $(\text{ord}(a)/p) \cdot a_1 \neq 0$  and thus

$$(8) \quad (\text{ord}(a)/p) \in \mathcal{M}(\text{PA}^-(\mathbb{K}))[p] \setminus T\mathcal{M}(\text{PA}^-(\mathbb{K}))[p].$$

We let  $o := o_T(a) \geq 0$  be such that  $a \in (T^o) \cdot \text{PA}^-(\mathbb{K}) \setminus (T^{o+1}) \cdot \text{PA}^-(\mathbb{K})$ .

In the second Chapter, we build a Thaine shift with respect to a prime  $\mathfrak{q} \in a_m$  and a lift  $b$  to  $a$  and derive the main cohomological properties of the shifted extension. The most important facts are the decomposition  $Tx \in \mathcal{D}(\text{PA}^-(\mathbb{L}))$  for all  $x \in \text{PA}^-(\mathbb{L})$  with  $\ell(x) \leq p \cdot \text{ord}(\mathcal{M}(\text{PA}^-(\mathbb{K})))$  and vanishing of the Tate cohomology  $\widehat{H}^0(F, \text{PA}^-(\mathbb{L}))$ . Based on this and the fact that  $Tb$  is decomposed while  $sb_m = 0$ , as the class of a ramified ideal, we obtain in Chapter 3 a sequence of algebraic consequences which eventually

lead to the fact that  $Ta = \mathcal{N}(b) \in \omega_m \mathcal{M}({}^pA^-(K))$ ; since this holds for arbitrary choices of  $m$ , independently of  $a$ , we obtain a contradiction to (8), which proves the Iwasawa conjecture.

The paper is written so that the main ideas of the proof can be presented in the main part of the text, leading in an efficient way to the final proof. The technical details and results are deduced with a richness of detail, in the appendices.

## 2. THAINE SHIFT AND PROOF OF THE MAIN THEOREM

We have selected in the first Chapter a base field  $\mathbb{K}$  which is CM and endowed with a list of properties. Consider the  $\mathbb{F}_p[[T]]$ -module  $P := {}^pA^-(\mathbb{K})/(p)$  and let  $\pi : {}^pA^-(\mathbb{K}) \rightarrow P$  be the natural projection. Then for any  $a \in \mathcal{M}({}^pA^-(\mathbb{K})) \setminus p \cdot {}^pA^-(\mathbb{K})$  there is some integer  $o_T(a) \geq 0$  such that the image  $\pi(a) \in {}^pA^-(\mathbb{K})/(p)$  verifies  $\pi(a) \in T^{o_T(a)}\pi({}^pA^-(\mathbb{K}))$ . We choose  $a \in \mathcal{M}({}^pA^-(\mathbb{K})) \setminus p \cdot {}^pA^-(\mathbb{K})$  with the minimal value of  $o_T(a)$ ; let  $m > \kappa(\mathbb{K})$  be such that

$$(9) \quad \deg(\omega_m) > 2(o_T(a) + 1)$$

and  $\mathfrak{q} \in a_m$  be a totally split prime. Let  $q \subset \mathbb{N}$  be the rational prime below  $\mathfrak{q}$ ; since  $\mathbb{K}$  contains the  $p^m$ -th roots of unity, it follows that  $q \equiv 1 \pmod{p^m}$ . We let  $\mathbb{L} = \mathbb{K} \cdot \mathbb{F}; \mathbb{L}_\infty = \mathbb{K}_\infty \cdot \mathbb{F}$  be the Thaine shift induced by  $\mathfrak{q}$ , as described in the section §1.2 and let  $b \in {}^pA^-(\mathbb{L})$  be a Thaine lift of  $a$ .

For  $C$  some  $\mathbb{Z}_p[s]$ -module, we use the Tate cohomologies associated to  $C$ , defined by

$$(10) \quad \begin{aligned} \hat{H}^0(F, C) &= \text{Ker}(s : C \rightarrow C)/(\mathcal{N}C), \\ \hat{H}^1(F, C) &= \text{Ker}(\mathcal{N} : C \rightarrow C)/(sC). \end{aligned}$$

The notation introduced here will be kept throughout the paper.

Let  $B'_n \subset A^-(\mathbb{K}_n)$  be the submodule spanned by the classes of primes that ramify in  $\mathbb{L}_n/\mathbb{K}_n$ . By choice of  $\mathbb{L}$ , these are the primes above  $q$  and consequently  $B'_n = \iota_{m,n}(B'_m)$  for all  $n > m$ . Here  $\iota_{m,n} : A^-(\mathbb{L}_m) \rightarrow A^-(\mathbb{L}_n)$  is the natural lift map. We let  $p^v$  be the exponent of  $B'_m$ , so  $p^v B'_n = 0$  for all  $n \geq m$ .

Since  $B'_n$  is constant up to isomorphism for all  $n > m$ , the vanishing of  $\hat{H}^0(F, {}^pA^-(\mathbb{L}))$  is a straight forward consequence of :

### Lemma 2.1.

$$(11) \quad \text{Ker}(s : {}^pA^-(\mathbb{L}) \rightarrow {}^pA^-(\mathbb{L})) = \iota({}^pA^-(\mathbb{K}))$$

In particular,  $\hat{H}^0(F, {}^pA^-(\mathbb{L})) = 0$ .

*Proof.* Consider  $x = (x_n)_{n \in \mathbb{N}} \in \text{Ker}(s : {}^pA^-(\mathbb{L}) \rightarrow {}^pA^-(\mathbb{L}))$  and let  $N > m + b$ . Let  $\mathfrak{X} \in x_N$  be a prime: then  $(\mathfrak{X}^{s(1-j)}) = (\xi^{1-j})$ , for some  $\xi \in \mathbb{L}_N$  and  $\mathcal{N}(\xi^{1-j}) \in \mu(\mathbb{K}_N)$ . Since  $\mathfrak{q}_m$  is inert in  $\mathbb{K}_N/\mathbb{K}_m$ , we have  $\mathcal{N}(\mathbb{L}_N) \cap \mu(\mathbb{K}_N) \subset \mu(\mathbb{K}_N)^p$ . We may thus assume, after eventually modifying  $\xi$  by a root of



unity, that  $\mathcal{N}(\xi^{1-j}) = 1$ . Hilbert's Theorem 90 implies that there is some  $\alpha \in \mathbb{L}_N$  such that

$$\mathfrak{X}^{s(1-j)} = (\xi^{1-j})^s = (\alpha^{1-j})^s \Rightarrow (\mathfrak{X}/(\alpha))^{(1-j)s} = (1).$$

The ideal  $\mathfrak{Y} := (\mathfrak{X}/(\alpha))^{1-j} \in x_N^2$  verifies  $\mathfrak{Y}^s = (1)$ . If  $x_N \notin \iota({}^p\mathcal{A}^-(\mathbb{K}))$ , then  $\mathfrak{Y}$  must be a product of ramified primes, so  $x_N^2 \in B'_N + \iota_{\mathbb{K}_N, \mathbb{L}_N}(\mathcal{A}_N^-(\mathbb{K}))$ . Recall that  $B'_N = \iota_{m,N}(B'_m)$  is spanned by the classes of the ramified primes and  $p^v B'_N = 0$ . In particular  $x_N^2 \in B'_N + \iota_{\mathbb{K}_N, \mathbb{L}_N}(\mathcal{A}_N^-(\mathbb{K}))$  and  $B'_N = \iota_{m,N}(B'_m)$  imply that

$$x_{N-v} = \mathbf{N}_{N, N-v}(x_N) \in \iota_{\mathbb{K}_{N-v}, \mathbb{L}_{N-v}}(\mathcal{A}^-(\mathbb{K}_{N-v})) + B'_m{}^{p^v} = \iota_{\mathbb{K}_{N-v}, \mathbb{L}_{N-v}}(\mathcal{A}^-(\mathbb{K}_{N-v})).$$

This happens for all  $N > m + v$ , so  $x \in \iota_{\mathbb{K}, \mathbb{L}}({}^p\mathcal{A}^-(\mathbb{K}))$ , as claimed.  $\square$

Note that at finite levels we have

$$\hat{H}^0(F, \mathcal{A}_n^-(\mathbb{L})) \cong B'_n / (B'_n \cap \iota_{\mathbb{L}/\mathbb{K}}(\mathcal{A}_n^-(\mathbb{K}))) \neq 0.$$

The Herbrand quotient of finite groups is trivial, so  $|\hat{H}^0(F, \mathcal{A}_n^-(\mathbb{L}))| = |\hat{H}^1(F, \mathcal{A}_n^-(\mathbb{L}))|$ . In the projective limit however,  $\hat{H}^1(F, {}^p\mathcal{A}^-(\mathbb{L})) \neq 0$  so equality is not maintained. In the Appendix §3.3, we prove though the following result:

**Lemma 2.2.** *Suppose that  $v \in \mathcal{M}({}^p\mathcal{A}^-(\mathbb{L}))$  has non trivial image in the Tate group  $\hat{H}^1(F, (\mathcal{M}({}^p\mathcal{A}^-(\mathbb{L}))))$ . Then either  $\text{ord}(v) > p \exp(\mathcal{M}({}^p\mathcal{A}^-(\mathbb{K})))$  or  $T^2v \in s^p\mathcal{A}^-(\mathbb{L})$ .*

We turn now our attention to the decomposition of the Thaine lift  $b$ ; we prove in the Appendix 3.3 the following

**Proposition 2.3.** *Let  $F = \langle \nu \rangle$  be a cyclic group of order  $p$  acting on the  $p$ -abelian group  $B$  and let  $x \in B$  be such that  $y = \mathcal{N}(x)$  has order  $\text{ord}(y) = q = p^l > p$ . Then  $\text{ord}(x) \leq pq$ .*

It implies

**Corollary 2.4.** *Let  $y \in {}^p\mathcal{A}^-(\mathbb{L})$  be such that  $\ell(\mathcal{N}(y)) = q > p$ . Then  $q \leq \ell(y) \leq pq$  and  $Ty \in \mathcal{D}({}^p\mathcal{A}^-(\mathbb{L}))$ . All these facts hold in particular for any Thaine lift  $b$ . In this case, one has additionally  $\text{ord}(sb) \leq \text{ord}(b) \leq pq$ .*

*Proof.* Let  $f(T) \in \mathbb{Z}_p[T]$  be a distinguished polynomial that annihilates  $p^{\ell(y)}y \in \mathcal{L}({}^p\mathcal{A}^-(\mathbb{L}))$  and let  $\beta = f(T)y$ . Then  $\alpha := \mathcal{N}(y)$  has the order  $q > p$ , by hypothesis, so we may apply the Proposition 2.3. This implies that  $\text{ord}(\beta) \leq pq$  and thus  $\ell(y) \leq pq \leq p \exp(\mathcal{M}({}^p\mathcal{A}^-(\mathbb{K})))$ , by definition of this order. The first claim in Remark 1.4 implies that  $Ty \in \mathcal{D}({}^p\mathcal{A}^-(\mathbb{L}))$ . Since  $\text{ord}(a) = \text{ord}(\mathcal{N}(b)) > p$  by choice of  $a$ , the statement applies in particular to any Thaine lift  $b$ . In this case, we know that  $pb_m = a_m$  and  $\text{ord}(a_m) = q$ , hence  $\text{ord}(b) \geq \text{ord}(b_m) = pq$ , hence  $\text{ord}(b) = pq$ . We obviously have  $\text{ord}(sb) \leq \text{ord}(b)$ .  $\square$

**2.1. The vanishing of  $\mu$ .** Since  $sb_m = 0$ , the Theorem VI of Iwasawa [5] implies that there is some  $c \in {}^p\mathcal{A}^-(\mathbb{L})$  such that  $sb = \nu_{m,1}c$ . Then  $\nu_{m,1}(\mathcal{N}(c)) = 0$  and the Fact 3.4 in the Appendix implies that  $\mathcal{N}(c) = 0$ . Moreover,

$$p^{\ell(b)}c = p^{\ell(b)}(\nu_{m,1}sb) \in \mathcal{L}({}^p\mathcal{A}^-(\mathbb{L}))$$

so, by Corollary 2.4,  $\ell(c) \leq pq$  too, and thus  $Tc \in \mathcal{D}({}^p\mathcal{A}^-(\mathbb{L}))$ . Let  $Tc = c_\lambda + c_\mu$ . Since  $\mathcal{L}({}^p\mathcal{A}^-(\mathbb{K})) \cap \mathcal{M}({}^p\mathcal{A}^-(\mathbb{K})) = 0$ , it follows that  $\mathcal{N}(c_\lambda) = \mathcal{N}(c_\mu) = 0$ , individually. We have, by comparing parts,  $sb_\mu = \nu_{m,1}c_\mu$ , so  $pq \cdot \nu_{m,1}c_\mu = 0$ , and since  $\mathcal{M}({}^p\mathcal{A}^-(\mathbb{L}))$  is rigid, (3) implies that  $\text{ord}(c_\mu) \leq pq$ . We may thus apply Lemma 2.2, which implies that  $T^2c_\mu \in s\mathcal{M}({}^p\mathcal{A}^-(\mathbb{L}))$ , say  $T^2c_\mu = sx$ ; then  $s(T^2b_\mu - \nu_{m,1}x) = 0$  and Lemma 2.1 implies that  $T^2b_\mu = \nu_{m,1}x + z$ ,  $z \in \iota({}^p\mathcal{A}^-(\mathbb{L}))$ . By taking norms we obtain  $T^3a = \nu_{m,1}x + pz$ . This implies  $\text{ord}_T(a) + 3 \geq \deg(\nu_{m,1})$  and we can choose  $m$  large enough, to obtain a contradiction. This confirms the claim of the Main Theorem.

### 3. APPENDIX

In the Appendix, unless otherwise specified, the notation used in the various Facts and Lemmata is the one used in the section where these are invoked in the text. In the next section we provide a list of disparate, useful facts:

**3.1. Auxiliary facts.** We start by proving that if  $\mu > 0$  for some number field, then it is also non - trivial for finite extensions thereof.

**Fact 3.1.** *Let  $K$  be a number field for which  $\mu(K) \neq 0$  and  $L/K$  be a finite extension, which is Galois over  $K$ . Then  $\mu(L) \neq 0$ .*

*Proof.* If  $M \subset L$  has degree coprime to  $p$ , then  $\text{Ker}(\iota : A(K) \rightarrow A(M)) = 0$ , so we may reduce the proof to the case of a cyclic Kummer extension of degree  $p$ . Let  $M = L^{\text{Gal}(L/K)_p}$  be the fixed field of some  $p$ -Sylow subgroup of  $\text{Gal}(L/K)$ . Then  $p$  does not divide  $[M : K]$ , so  $\text{Ker}(\iota : A^-(K) \rightarrow A^-(M)) = 0$ , and thus  $\mu(M) \neq 0$ . We may assume without loss of generality, that  $M$  contains the  $p$ -th roots of unity. Since  $p$ -Sylow groups are solvable, the extension  $L/M$  arises as a sequence of cyclic Kummer extensions of degree  $p$ . It will thus suffice to consider the case in which  $k$  is a number field with  $\mu \neq 0$  and containing the  $p$ -th roots of unity and  $k' = k[a^{1/p}]$  is a cyclic Kummer extension of degree  $p$ . We claim that under these premises,  $\mu(k') \neq 0$ . Let  $k_n \subset k_\infty$  and  $k'_n \subset k'_\infty$  be the intermediate fields of the cyclotomic  $\mathbb{Z}_p$ -extensions, let  $\nu$  generate  $\text{Gal}(k'/k)$ . Let  $F/k_\infty$  be an abelian unramified extension with  $\text{Gal}(F/k_\infty) \cong \mathbb{F}_p[[T]]$ ; such an extension must exist, as a consequence of  $\mu > 0$ . There is thus for each  $n > 0$  a  $\delta_n \in k_n^\times$  such that  $F_n = k_n \left[ \delta_n^{\mathbb{F}_p[[T]]/p} \right]$  is an unramified extension with galois group  $G_n = \text{Gal}(F_n/k_n)$  of  $p$ -rank  $r_n := p\text{-rk}(G_n) > p^{n-c}$  for some  $c \geq 0$ . We define  $F'_n = F_n[a^{1/p}]$  and let  $\overline{F}'_n \supset F'_n$  be the maximal subextension which is unramified over  $k'_n$ . We have  $\overline{F}'_n \supseteq F_n$  and thus  $p\text{-rk}(\text{Gal}(\overline{F}'_n/k'_n)) \geq$

$p\text{-rk}(\text{Gal}(F_n/k_n)) \rightarrow \infty$ . Consequently,  $k'_\infty$  has an unramified elementary  $p$ -abelian extension of infinite rank, and thus  $\mu(k') > 0$ , which completes the proof.  $\square$

**Fact 3.2.** *Let  $\mathbb{K}$  be a CM extension with  $\mu > 0$ . Then it is possible to build a further CM extension  $\mathbb{L}/\mathbb{K}$  with  $\exp(\mathcal{M}^-(\mathbb{L})) > p^2$ .*

*Proof.* We have shown in Fact 3.1 that we may assume that  $\mu_{p^3} \subset \mathbb{K}$ . Let  $a = (a_n)_{n \in \mathbb{N}} \in \mathcal{M}^-(\mathbb{K})$  and  $\mathfrak{q} \in a_2$ , with  $a_2 \neq 0$ , be a totally split prime which is inert in  $\mathbb{K}_\infty/\mathbb{K}_2$ . Let  $\mathbb{L}/\mathbb{K}_2$  be the inert Thaine shift of degree  $p^2$  induced by  $\mathfrak{q}$ , let  $b_2 = [\mathfrak{Q}^{(1-j)/2}]$  be the class of the ramified prime of  $\mathbb{L}$  above  $\mathfrak{q}$  and  $b = (b_m)_{m \in \mathbb{N}}$  be a sequence through that extends  $b_2$ , such that  $N_{\mathbb{L}/\mathbb{K}}(b) = a$ . Then  $b \notin \mathcal{L}^-(\mathbb{L})$  and there is some polynomial  $f(T) \in \mathbb{Z}_p[T]$  such that  $f(T)b \in \mathcal{M}^-(\mathbb{L})$ , while  $N_{\mathbb{L}/\mathbb{K}}(f(T)b) = f(T)a$ . The capitulation kernel  $\text{Ker}(\iota : A^-(\mathbb{K}_n) \rightarrow A^-(\mathbb{L}))$  is trivial and consequently  $\text{ord}(Tf(T)b) \geq p^2 \text{ord}(a)$ ; hence  $\exp(M^-(\mathbb{L})) > p^2$ . Thus  $\mathbb{L}$  verifies the claimed properties.  $\square$

The following fact was proved by Sands in [6]:

**Fact 3.3.** *Let  $\mathbb{L}/\mathbb{K}$  be a  $\mathbb{Z}_p$ -extension of number fields in which all the primes above  $p$  are completely ramified. If  $F(T) \in \mathbb{Z}_p[T]$  is the minimal annihilator polynomial of  $\mathcal{L}(\mathbb{L})$ , then  $(F, \nu_{n,1}) = 1$  for all  $n > 1$ .*

As a consequence,

**Corollary 3.4.** *Let  $\mathbb{K}$  be a CM field and suppose that  $x \in {}^pA^-(\mathbb{K})$  verifies  $\nu_{n,1}x = 0$ . Then  $x = 0$ .*

*Proof.* Let  $q$  be the exponent of the  $\mathbb{Z}_p$ -torsion of  ${}^pA^-(\mathbb{K})$ . It follows then that  $qx \in \mathcal{L}({}^pA^-(\mathbb{K}))$  is annihilated by  $\nu_{n,1}$ , so the Fact 3.3 implies that  $qx = 0$  and thus  $x \in \mathcal{M}({}^pA^-(\mathbb{K}))$ . Since  $\nu_{m,1}x = 0$  it follows that  $x = 0$ , as claimed.  $\square$

**3.2. Applications of the Tchebotarew Theorem.** We prove the existence of Thaine lifts.

**Lemma 3.5.** *Let  $\mathbb{K}$  be a CM field and  $a = (a_n)_{n \in \mathbb{N}} \in {}^pA^-(\mathbb{K})$  and  $\mathbb{L} = \mathbb{K} \cdot \mathbb{F}$  be a Thaine shift induced by a split prime  $\mathfrak{q} \in a_m$ . Then there is a Thaine lift  $b = (b_n)_{n \in \mathbb{N}} \in {}^pA^-(\mathbb{L})$  with the properties that  $\mathcal{N}(b) = a$  and  $b_m$  is the class of the ramified prime  $\mathfrak{Q} \subset \mathbb{L}$  above  $\mathfrak{q}$ .*

*Proof.* We prove by induction that for each  $n \geq 1$  there is a class  $b_n \in A_n^-(\mathbb{L})$  with  $\mathcal{N}(b_n) = a_n$  and  $N_{n,n-1}(b_n) = b_{n-1}$ . The claim holds for  $n \leq m$  by definition. Assume that it holds for  $n \geq m$  and consider minus parts of the maximal  $p$ -abelian unramified extensions

$$\mathbb{H}(\mathbb{K}_n)^-/\mathbb{K}_n, \mathbb{H}^-(\mathbb{L}_n)/\mathbb{L}_n, \mathbb{H}^-(\mathbb{L}_{n+1})/\mathbb{L}_{n+1}, \quad \text{and} \quad \mathbb{H}^-(\mathbb{K}_{n+1})/\mathbb{K}_{n+1}.$$

For  $\mathbb{H}' \in \{\mathbb{H}^-(\mathbb{K}_n), \mathbb{H}^-(\mathbb{L}_n), \mathbb{H}^-(\mathbb{K}_{n+1})\}$ , we obviously have  $\mathbb{H}' \cdot \mathbb{L}_{n+1} \subset \mathbb{H}^-(\mathbb{L}_{n+1})$ . For some unramified Kummer extension  $M/K$  we denote by

$\varphi(x) = \left(\frac{M/K}{x}\right)$  the Artin symbol of a class or of an ideal. The induction hypothesis implies that both  $\varphi(b_n)$  and  $\varphi(a_{n+1})$  restrict to  $\varphi(a_n) \in \text{Gal}(\mathbb{H}^-(\mathbb{K}_n)/\mathbb{K}_n)$ , and since  $A_{n+1}^-(\mathbb{K})$  and  $A_n^-(\mathbb{L})$  surject by the respective norms to  $A_n^-(\mathbb{K}_n)$ , it follows that

$$\mathbb{L}_{n+1}\mathbb{H}(\mathbb{K}_{n+1}) \cap \mathbb{L}_{n+1}\mathbb{H}^-(\mathbb{L}_n) = \mathbb{L}_{n+1}\mathbb{H}^-(\mathbb{K}_n).$$

There is thus some automorphism  $x \in \text{Gal}(\mathbb{H}^-(\mathbb{L}_{n+1})/\mathbb{L}_{n+1})$ , such that

$$x|_{\text{Gal}(\mathbb{H}^-(\mathbb{L}_n)/\mathbb{L}_n)} = \varphi(b_n) \quad \text{and} \quad x|_{\text{Gal}(\mathbb{H}^-(\mathbb{K}_{n+1})/\mathbb{K}_{n+1})} = \varphi(a_{n+1}).$$

By Tchebotarew, there are infinitely many totally split primes  $\mathfrak{R} \subset \mathbb{L}_{n+1}$  with Artin symbol  $\left(\frac{\mathbb{H}^-(\mathbb{L}_{n+1})/\mathbb{L}_{n+1}}{\mathfrak{R}}\right) = x$ , and by letting  $b_{n+1} = [\mathfrak{R}]$  for such a prime, we have  $\mathcal{N}(b_{n+1}) = a_{n+1}$  and  $N_{n+1,n}(b_{n+1}) = b_n$ . We obtain by induction a norm coherent sequence  $b = (b_n)_{n \in \mathbb{N}}$  which verifies  $b_m = [\mathfrak{Q}]$  and  $\mathcal{N}(b) = a$ , and this completes the proof.  $\square$

**3.3. Proof of Proposition 2.3 and related results.** Let the notations be like in the statement of the Proposition and let  $q = p^k = \text{ord}(y)$  and  $r \geq q$  be the order of  $x$  and  $\mathbf{R} = \mathbb{Z}/(r \cdot \mathbb{Z})[s]$ . Then  $\mathbf{R}$  has maximal ideal  $(p, s)$  and  $s$  is nilpotent. By definition,  $\mathbf{R}$  acts on  $x$  and we consider the modules  $X = \mathbf{R}x$  and  $Y = \mathbb{Z}/(q' \cdot \mathbb{Z})y \subset X$ .

With these notations, we are going to prove that  $pqx = 0$ . We start by proving

**Lemma 3.6.** *Under the given assumptions on  $x, y$  and with the notations above,*

$$\hat{H}^0(F, X) = 0,$$

*or  $qpx = 0$ . Moreover,  $sX \cap Y \cong \mathbb{F}_p \subset X[s, p]$  in all cases, while if  $\hat{H}^0(F, X) \neq 0$ , the bi-torsion  $X[s, p] \cong \mathbb{F}_p^2$  and  $q'x \in X[s, p] \setminus sX$ .*

*Proof.* Let  $N$  be a sufficiently large integer, such that  $p^N x = 0$  and let  $R = \mathbb{Z}/(p^N \cdot \mathbb{Z})$ , so  $X$  is an  $R[s]$ -module. We note that in  $R' := R/(\mathcal{N})$  the maximal ideal is  $(s)$ , thus a principal nilpotent ideal, as one can verify from the definition of the norm  $\mathcal{N} = \sum_{i=0}^{p-1} \nu^i$ . Indeed, in  $R$  we have the identity  $p = s^{p-1} \cdot v(s) \bmod \mathcal{N}$ ,  $v(s) \in R^\times$ , so the image of  $p$  in  $R'$  is a power of  $s$ , hence the claim. As a consequence, in any finite cyclic  $R'$ -module  $M$  we have  $M[p, s] \cong \mathbb{F}_p$ .

We consider the two modules  $N_1 = \mathbb{Z}_p x$  and  $N_2 = R s x$ , such that  $X = N_1 + N_2$ , and only  $N_2$  is an  $R$ -module, and in fact an  $R'$ -module, since it is annihilated by  $\mathcal{N}^1$ . Note that  $qN_1$  is also annihilated by the norm; this covers also the case when  $qx = 0$ , so  $qN_1 = 0$ . Since  $X = N_1 + N_2$ , it follows that  $X[p] = N_1[p] + N_2[p]$ .

We always have that  $N_2[s] \cong \mathbb{F}_p$ , since  $N_2$  is an  $R'$ -module and  $(s)$  is the maximal ideal of  $R'$ . Let  $K := X[s] = \text{Ker}(s : X \rightarrow X)$  and assume that

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<sup>1</sup>This proof is partially inspired by some results in the Ph. D. Thesis of Tobias Bembom [1]

$H_0 := \hat{H}^0(F, X) \neq 0$ ; then the bi-torsion  $X[p, s] = X[s][p]$  is not  $\mathbb{F}_p$ -cyclic. It follows that  $X[p, s] \not\cong N_1 \cap X[p, s] \cong \mathbb{F}_p$ . Let  $r = qp^e = \text{ord}(x)$ , with  $e \geq 0$ . Note that

$$y_0 := (q/p)y = qxu(s) + (q/p)s^{p-1} \in X[p, s],$$

for all values of  $e$ . If  $e = 0$ , then  $y_0 = (q/p)s^{p-1} \in N_2[p, s]^\times$ . But then  $(q/p)sx \neq 0$  and thus  $N_1[p, s] = 0$  so  $\hat{H}(F, X) = 0$  in this case. Assume that  $e > 0$  and  $y_0 \in N_2$ . Since  $y_0 = qx + (q/p)f(s)(sx)$  and the last term,  $y_1 := (q/p)f(s)(sx) \in N_2$ , it follows that  $qx = y_0 - y_1 \in N_2$ , so  $\hat{H}(F, X) = 0$  in this case too. It remains that if  $\hat{H}(F, X) \neq 0$ , then  $y_0 \in N_1 \setminus N_2$  and  $e > 0$ .

In this case  $N_1 \cap N_2 = \emptyset$  and  $X = N_1 \oplus N_2$ . We let  $(q/p)(p^e x - cy) = 0$ , with  $(c, p) = 1$ . Then there is some  $w \in X[q/p]$  with  $p^e x = cy + w$ , and the decomposition  $X = N_1 \oplus N_2$  induces a decomposition of the  $q/p$ -torsions, so we may write  $w = a(q/p) + bv(s)s^N x$ , with  $N > 0$  and  $a, b \in \{0, 1, \dots, p-1\}$ ,  $v(s) \in R'^\times$ . Taking norms in the identity  $(p^e - a(q/p))x = cy + bv(s)s^N x$  we obtain

$$py \cdot (c - p^{e-1} + a(q/p^2)) = 0.$$

We have assumed  $q > p^2$ , so for  $e > 1$  the cofactor of  $py$  is a unit, which implies  $py = 0$ , in contradiction with  $\text{ord}(y) = q > p^2$ . Therefore  $e = 1$  and

$$(12) \quad y = px + v(s)s^{p-1}x, \quad \text{and} \quad (q/p)s^{p-1}x = 0.$$

Consequently, if  $\hat{H}(F, X) \neq 0$ , then  $\text{ord}(x) = qp$  and there is an  $N < (p-1)k$  such that  $N_2[s] = \mathbb{F}_p \cdot (s^N x)$ . This completes the proof.  $\square$

We can now assume that  $\hat{H}^0(F, X) = 0$  and since  $X$  is finite, the Herbrand quotient vanishes and  $\hat{H}^1(F, X) = 0$ . There is thus an exact sequence of  $\mathbb{Z}_p[s]$ -modules

$$0 \rightarrow X[p] \rightarrow X \rightarrow X \rightarrow X/pX \rightarrow 0,$$

in which  $X/pX$  is cyclic generated by  $x$  and the arrow  $X \rightarrow X$  is the multiplication by  $p$  map  $\cdot p$ .

Since  $\hat{H}^0(F, X) = 0$ , it follows that  $\text{Ker}(s : X \rightarrow X)[p] \cong \mathbb{F}_p$  and  $X[p]$  is  $\mathbb{F}_p[s]$ -cyclic, as is  $X/pX$ . Using this observation, we provide now the proof of the Proposition.

*Proof.* There is some distinguished polynomial  $\phi \in \mathbf{R}$  with  $\phi(s)X = 0$  and  $\deg(\phi) = p\text{-rk}(X)$ . Indeed, let  $d = p\text{-rk}(X)$  and  $\bar{x} \in X/pX$  be the image of  $x$ , so  $(s^i \bar{x})_{i=0, d-1}$  have independent images in  $X/pX$ , by definition of the rank and span  $X$ , as a consequence of the Nakayama Lemma. Therefore  $s^d x \in pX$  and there is a monic distinguished annihilator polynomial

$$\phi(s) = s^d - p^e h(s)$$

of  $X$ , with  $e \geq 0$  and  $h$  a polynomial of  $\deg(h) < \deg(\phi) \leq p$ , which is not  $p$ -divisible. Note that, by minimality of  $d$ , the case  $e = 0$  only occurs

if  $\phi(s) = s^d$ . We shall distinguish several cases depending on the degree  $d = \deg(\phi)$  and on properties of  $h(s)$ .

We consider the cases in which  $d < p$  first. Assume that  $s^c x = 0$  for some  $c < p$ , so  $h = 0$ . Upon multiplication with  $s^{p-1-c}$  we obtain  $s^{p-1}X = 0$ . Thus  $\mathcal{N}(x) = y = (s^{p-1} + pu(s))x = pu(s)x$ , and  $yu^{-1}(s) = y = px$ , which settles this case.

We can assume now that  $\phi(s) = s^d - p^e h(s)$  and  $e > 0$ , with  $h(s)$  a non trivial distinguished polynomial of degree  $\deg(h) < d < p$ . As a consequence, if  $d < p - 1$ , then

$$\begin{aligned} y &= \mathcal{N}(x) = pu(s)x + s^{p-1}x = p(u(s) + s^{p-1-d}p^{e-1}h(s))x \\ &= pv(s)x, \quad v(s) = u(s) + s^{p-1-d}p^{e-1}h(s) \in \mathbf{R}^\times. \end{aligned}$$

Hence  $y = v^{-1}(s)z = px$ , which confirms the claim in this case.

If  $d = p - 1$ , then  $s^{p-1}x = p \cdot (p^{e-1}h(s))x$  and thus

$$y = (pu(s) + s^{p-1})x = p(u(s) + p^{e-1}h(s))x;$$

if the expression in the brackets is a unit, we may conclude like before. Otherwise,  $e = 1$  and  $h(s) = -1 + sh_1(s)$ , and thus  $y \in sX$ , so  $\mathcal{N}(y) = py = 0$ , in contradiction with the assumption<sup>2</sup> that  $\text{ord}(y) > p$ .

The case  $d = p$  is more involved. We claim that  $\text{ord}(x) < p^2 \text{ord}(y)$ . Assume that this is not the case. Since  $d = p$ , we have  $p\text{-rk}(X) = p$  and thus  $s^{p-1}x = y - pxu(s) \notin pX$ . In particular  $y \notin pX$ . Let  $q = p^k = \text{ord}(y)$  and assume that  $\text{ord}(x) = p^{e+k} = p^e \cdot \text{ord}(y)$ ,  $e > 1$ . We note that  $\text{ord}(s^{p-1}x) = p^{e-1}q$ , since  $s^{p-1}u^{-1}(s)x = -px + y$  has annihilator  $p^{e-1}q$ . Consider the generators  $s^j x$  of  $X$ ; there is an integer  $j$  in the interval  $0 \leq j < p - 1$ , such that

$$\text{ord}(s^j x) = \text{ord}(x) > \text{ord}(s^{j+1}x) = \text{ord}(s^{p-1}x) = \text{ord}(x)/p.$$

Recall from Lemma 3.6, that we are in the case when  $X[p]$  is a cyclic  $\mathbb{F}_p[s]$  module of dimension  $p$  as an  $\mathbb{F}_p$ -vector space, and  $\widehat{H}^0(F, X) = 0$ . Let

$$\begin{aligned} \mathcal{F}_0 &:= \{qp^{e-1}s^i x : i = 0, 1, \dots, j\} \subset X[p], \\ \mathcal{F}_1 &:= \{qp^{e-2}s^{j+i}x : i = 1, 2, \dots, p - j - 1\} \subset X[p], \end{aligned}$$

and  $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ . Then  $\mathcal{F}_i \subset X[p]$  are  $\mathbb{F}_p$ -bases of some cyclic  $\mathbb{F}_p[s]$  submodules  $F_0, F_1 \subset X[p]$  with  $\dim_{\mathbb{F}_p}(F_0) \leq j + 1$  and  $\dim_{\mathbb{F}_p}(F_1) \leq p - (j + 1)$ . We claim that  $X[p] = F_0 \oplus F_1 = 0$ . For each  $z \in X[p]$  there is some maximal  $z' \in X$  – thus  $z'$  having non-trivial image  $0 \neq \overline{z'} \in X/pX$  – and such that  $z = q'z'$  for some  $q' \in p^{\mathbb{N}}$ . Since the generators of  $X/pX$  are mapped this way to  $F = F_0 + F_1$ , which is an  $\mathbb{F}_p$ -vector space, it follows by linearity that  $F = X[p]$ . Comparing dimensions, we find that  $F_0 \cap F_1 = 0$  so there is a direct sum  $F = F_0 \oplus F_1 = X[p]$ , as claimed.

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<sup>2</sup>At this point the assumption that  $\text{ord}(y) > p$  plays a crucial role, and if it were not to hold, modules such that  $\text{ord}(x)$  becomes arbitrarily large are conceivable

Note that

$$0 \neq (q/p)y = qx + (q/p)s^{p-1}u^{-1}(s)x \in X[p][s];$$

upon multiplication with  $p^{e-1} \geq p$  we obtain

$$\begin{aligned} 0 &= qp^{e-2}y = qp^{e-1}u(s)x + qp^{e-2}s^{p-1}x \\ (13) \quad &= qp^{e-1}(u_0(s) + u_1(s))x + qp^{e-2}s^{p-1}x, \end{aligned}$$

where  $u_0 \in F_0$  and  $u_1 \in F_1$  are the projections of the unit  $u(s)$ , thus

$$u_0(s) \equiv \frac{\sum_{i=0}^j s^i \binom{p}{i}}{p} \pmod{p}, \quad u_1(s) \equiv \frac{\sum_{i=j+1}^{p-2} s^i \binom{p}{i}}{p} \pmod{p}.$$

By definition of  $j$ , it follows that

$$f_0 := qp^{e-1}u_0(s)x \in F_0, \quad f_1 := s^{p-1}qp^{e-2}x \in F_1,$$

and  $qp^{e-1}u_1(s)x = 0$ . The identity in (13) becomes  $f_0 + f_1 = 0$ , and since  $F_0 \cap F_1 = 0$  and  $f_i \in F_i$ ,  $i = 0, 1$ , it follows that  $f_0 = f_1 = 0$ . However,  $f_1 = s^{p-1}qp^{e-2}x \in \mathcal{F}_1$  is a basis element which generates  $F_1[s]$ , so it cannot vanish. The contradiction obtained implies that we must have in this case also  $\text{ord}(x) \leq pq$ . Consequently,  $\text{ord}(x) \leq p \cdot \text{ord}(y)$  in all cases, which completes the proof of the Proposition.  $\square$

#### 3.4. The Norm Principle, ray class fields and proof of Lemma 2.2.

Let  $q$  be a rational prime with  $q \equiv 1 \pmod{p^m}$ . Let  $\mathbf{F} \subset \mathbb{Q}_q[\zeta_q]$  be the subfield of the (ramified)  $q$ -th cyclotomic extension, which has degree  $p$  over  $\mathbb{Q}_q$ . Thus  $\mathbf{F}$  is the completion at the unique ramified prime above  $q$  of the field  $\mathbb{F}$  defined in the text. The field  $\mathbf{F}$  is a tamely ramified extension of  $\mathbb{Q}_q$ , so class field theory implies that  $\text{Gal}(\mathbf{F}/\mathbb{Q}_q)$  is isomorphic to a quotient of order  $p$  of  $(\mathbb{Z}/q \cdot \mathbb{Z})^*$ , so letting  $S = ((\mathbb{Z}/q \cdot \mathbb{Z})^*)^p$ , we have  $\text{Gal}(\mathbf{F}/\mathbb{Q}_q) \cong (\mathbb{Z}/q \cdot \mathbb{Z})^*/S$ . Let thus  $r \in \mathbb{Z}$  be such that  $r^{(q-1)/p} \not\equiv 1 \pmod{q}$ : if  $g \in \mathbb{F}_q^\times$  generates the multiplicative group of the finite field with  $q$  elements and  $m = v_p(q-1)$ , then one can set  $r = g^{(q-1)/p^m} \pmod{q}$ . Let in addition  $\mathbf{K}_n/\mathbb{Q}_q$  be the  $p^n$ -th cyclotomic extension of  $\mathbb{Q}_q$ : under the given premises, we have for  $n > m$  the extension degree  $[\mathbf{K}_n : \mathbb{Q}_q] = p^{n-m}$  and  $\mathbf{K}_n = \mathbb{Q}_q[r^{1/p^{n-m}}]$ , while for  $n \leq m$  the extension is trivial. Letting  $r_n = r^{1/p^{n-m}} \in \mathbf{K}_n$ , and  $\mathbf{L}_n = \mathbf{K}_n \cdot \mathbf{F}$ , we deduce by class field theory that  $r_n$  generates  $\mathbf{K}_n^\times/\mathcal{N}(\mathbf{L}_n^\times)$ , under the natural projection. Indeed, the extension  $\mathbf{L}_n/\mathbf{K}_n$  is a ramified  $p$  extension, so the galois group must be a quotient of the roots of unity  $W(\mathbb{Z}_q)$ , hence the claim. We shall use these elementary observation in order to derive the structure of  $\widehat{H}^1(F, A^-(\mathbb{L}_n))$  in our usual setting and prove some necessary conditions for elements  $v_n \in A^-(\mathbb{L}_n)$  which verify  $0 \neq \beta(v_n) \in \widehat{H}^1(F, A^-(\mathbb{L}_n))$ , under the natural projection  $\beta : A^-(\mathbb{L}_n) \rightarrow \widehat{H}^1(F, A^-(\mathbb{L}_n))$ .

Let  $\mathbb{K}, \mathbb{L}, \mathbb{F}$ , etc. be the fields defined in the main part of the paper and let us denote by  $I(\mathbb{M})$  the ideals of some arbitrary number field, and  $P(\mathbb{M}) \subset I(\mathbb{M})$  the principal ideals. The maximal  $p$ -abelian unramified extension is  $\mathbb{H}(\mathbb{M})$  and the  $p$ -part of the ray class field to the ray  $\mathfrak{M}_q = q \cdot \mathcal{O}(\mathbb{M})$  will

be denoted by  $\mathbb{T}_q(\mathbb{M})$ . If  $\mathbb{M}$  is a CM field, then complex conjugation acts, inducing  $I^-(\mathbb{M})$ ,  $P^-(\mathbb{M})$  and  $\mathbb{H}^-, \mathbb{T}_q^-$ , in the natural way. In our context, we let in addition  $P_N^-(\mathbb{L}) := \mathcal{N}(P^-(\mathbb{L})) \subset P^-(\mathbb{K})$ . We let  $\Delta_n = \text{Gal}(\mathbb{K}_n/\mathbf{k})$  for arbitrary  $n$ . The following is an elementary result in the proof of the Hasse Norm Principle:

**Lemma 3.7.** *Let  $\mathbb{K}, \mathbb{L}$  and  $F$  be like above. Then*

$$(14) \hat{H}^{(1)}(F, A^-(\mathbb{L}_n)) \cong P^-(\mathbb{K}_n)/P_N^-(\mathbb{L}_n) \cong \mathbb{F}_p[\Delta_n], \quad \text{for all } n > 0,$$

*the isomorphism being one of cyclic  $\mathbb{F}_p[\Delta_n]$ -modules.*

*Proof.* Note that both modules in (14) are annihilated by  $p$ . In the case of  $P^-(\mathbb{K})/P_N^-(\mathbb{L})$ , this is a direct consequence of  $(P^-(\mathbb{K}))^p = \mathbf{N}_{\mathbb{L}/\mathbb{K}}(P^-(\mathbb{K})) \subset P_N^-(\mathbb{L})$ . Let  $\beta : A^-(\mathbb{L}) \rightarrow \hat{H}^{(1)}(F, A^-(\mathbb{L}))$  and  $\pi_N : P^-(\mathbb{K}) \rightarrow P^-(\mathbb{K})/P_N^-(\mathbb{L})$  denote the natural projections and let  $a \in \text{Ker}(\mathcal{N} : A^-(\mathbb{L}) \rightarrow A^-(\mathbb{L}))$ . Then  $pu(s)a = -s^{p-1}a$  and thus  $pa = -s^{p-1}u^{-1}(s)a$  and a fortiori  $\beta(pa) = 0$  for all  $a$ , so  $p\hat{H}^{(1)}(F, A^-(\mathbb{L})) = 0$ , thus confirming that  $\hat{H}^{(1)}(F, A^-(\mathbb{L}_n))$  is an  $\mathbb{F}_p$ -module too.

Let now  $\mathfrak{A} \in a$  be some ideal and  $(\alpha) = \mathcal{N}(\mathfrak{A})$ . The principal ideal  $\mathfrak{a} := (\alpha/\bar{\alpha}) \in P^-(\mathbb{K})$  has image  $\pi_N(\mathfrak{a}) \in P^-(\mathbb{K})/P_N^-(\mathbb{L})$  which depends on  $a$  but not on the choice of  $\mathfrak{A} \in a$ . This is easily seen by choosing a different ideal  $\mathfrak{B} = (x)\mathfrak{A} \in a$ : then  $\mathcal{N}(\mathfrak{B}^{1-j}) = \mathfrak{a} \cdot \mathcal{N}(x/\bar{x}) \in \mathfrak{a} \cdot P_N^-(\mathbb{L})$ , and  $\pi_N(\mathcal{N}(\mathfrak{B}^{1-j})) = \pi_N(\mathcal{N}(\mathfrak{A}^{1-j})) = \pi_N(\mathfrak{a})$  depends only on  $a$ . Suppose now that  $\mathfrak{a} \in P_N^-(\mathbb{L})$ , so  $\pi_N(\mathfrak{a}) = 1$ . Then there is some  $y \in \mathbb{L}^\times$  such that  $\mathcal{N}(\mathfrak{A}^{1-j}) = (\mathcal{N}(y)^{1-j})$  and thus  $\mathcal{N}(\mathfrak{A}/(y))^{1-j} = (1)$ . Since  $\hat{H}^{(1)}$  vanishes for ideals, it follows that there is a further ideal  $\mathfrak{X} \subset \mathbb{L}$  such that

$$\mathfrak{A}^{1-j} = ((y)\mathfrak{X}^s)^{1-j},$$

and thus  $a^2 \in (A^-(\mathbb{L}))^s$ . But then  $\beta(a) = 0$ . We have shown that there is a map  $\lambda : \hat{H}^{(1)}(F, A^-(\mathbb{L})) \rightarrow P^-(\mathbb{K})/P_N^-(\mathbb{L})$  defined by the sequence of associations  $\beta(a) \mapsto \mathfrak{a} \mapsto \pi_N(\mathfrak{a})$ , which is a well defined injective homomorphism of  $\mathbb{F}_p$ -modules.

In order to show that  $\lambda$  is an isomorphism, let  $\mathfrak{x} := (x/\bar{x}) \in P^-(\mathbb{K}) \setminus P_N^-(\mathbb{L})$  be a principal ideal that is not a norm from  $\mathbb{L}$ . Let the Artin symbol of  $x$  be  $\sigma = \left(\frac{\mathbb{L}/\mathbb{K}}{x}\right) \in \text{Gal}(\mathbb{L}/\mathbb{K})$ ; by definition,  $\mathbb{L}$  is also CM, so complex conjugation commutes with  $\sigma$  and we have

$$\left(\frac{\mathbb{L}/\mathbb{K}}{\bar{x}}\right) = \left(\frac{\mathbb{L}/\mathbb{K}}{x^j}\right) = \sigma^j = \sigma.$$

Consequently,  $\left(\frac{\mathbb{L}/\mathbb{K}}{(x/\bar{x})}\right) = 1$  – we may thus choose, by Tchebotarew, a principal prime  $(\rho) \subset \mathbb{K}$  with  $\rho \cong x/\bar{x} \pmod{q}$ , and which is split in  $\mathbb{L}/\mathbb{K}$ . Let  $\mathfrak{R} \subset \mathbb{L}$  be a prime above  $(\rho)$  and  $r := [\mathfrak{R}^{1-j}] \in (A^-(\mathbb{L}))$ . We claim that  $\beta(r) \neq 0$ ; assume not, so  $\mathfrak{R} = (y)\mathfrak{Y}^s$  for some  $y \in \mathbb{L}$  and  $\mathfrak{Y} \subset \mathbb{L}$  and thus  $\mathcal{N}(\mathfrak{R}^{1-j}) = (\mathcal{N}(y/\bar{y})) \cong \mathfrak{x} \pmod{P_N^-(\mathbb{L})}$ . Since  $(\mathcal{N}(y/\bar{y})) \in P_N^-(\mathbb{L})$  by definition, it follows that  $\mathfrak{x} = (x/\bar{x}) \in P_N^-(\mathbb{L})$ , which contradicts the choice of  $x$ .



It follows that  $\lambda$  is an isomorphism of  $\mathbb{F}_p[\Delta_m]$ -modules. The proof will be completed if we show that  $P^-(\mathbb{K})/P_N^-(\mathbb{L}) \cong \mathbb{F}_p[\Delta_m]$ . Since  $\Delta_m$  acts transitively on the pairs of complex conjugate primes above  $q$  in  $\mathbb{K}_n$ , one verifies that  $(\mathbb{K}_n^\times)^-/\mathcal{N}((\mathbb{L}_n^\times)^-) \cong \mathbb{F}_p[\Delta_m]$ . The field  $\mathbf{k}$  contains no  $p$ -th roots of unity, and thus  $\mathbf{N}_{\mathbb{K}_n/\mathbf{k}}(P^-(\mathbb{K}_n)/P_N^-(\mathbb{L}_n)) \neq 0$ . The isomorphism

$$P^-(\mathbb{K})/P_N^-(\mathbb{L}) \cong \mathbb{F}_p[\Delta_m]$$

follows, and this completes the proof.  $\square$

Consider the field  $\mathbb{T}'_n := \mathbb{T}_q^-(\mathbb{K}_n)$  defined as the minus  $p$ -part of the ray class field to the modulus  $Q_n := q\mathcal{O}(\mathbb{K}_n)$  – i.e., if  $\mathbf{T}$  is the full ray class field to  $Q_n$  and  $\mathbf{T}_p$  is the  $p$ -part of this field, thus fixed by all  $q'$ -Sylow subgroups of  $\text{Gal}(\mathbf{T}/\mathbb{K}_n)$ , for all primes  $q' \leq q$ , then  $\mathbb{T}'_n$  is the subfield of  $\mathbf{T}_p$  fixed by  $(\text{Gal}(\mathbf{T}_p/\mathbb{K}_n))^+$ . As mentioned above, the completions at individual primes  $\mathfrak{q}'$  above  $q$  are cyclic groups

$$\mathbb{T}'_{n,\mathfrak{q}'}/\mathbb{H}(\mathbb{K}_n)_{\mathfrak{q}'} \cong (\mathbb{F}_q[\zeta_{p^n}])^\times_p \cong C_{p^n}.$$

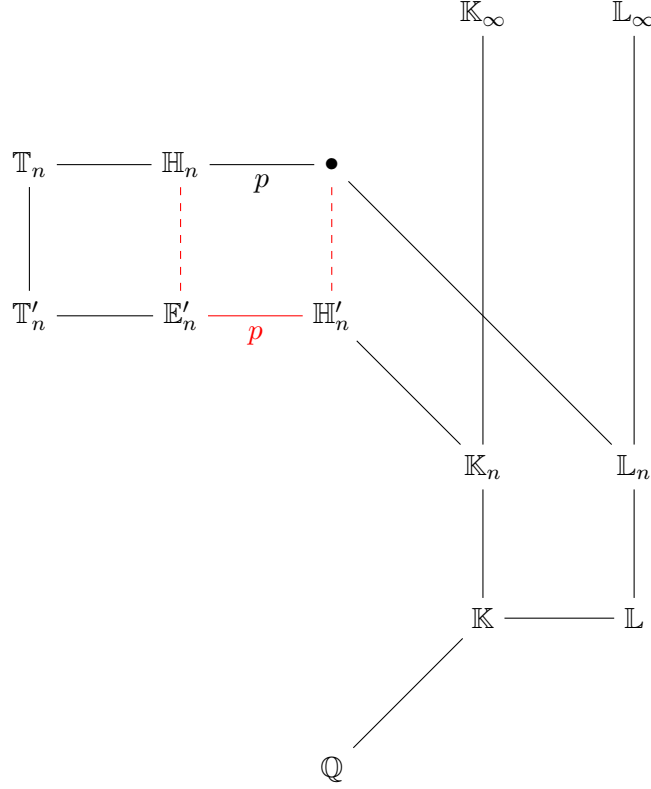
Consequently,

$$(15) \quad \text{Gal}(\mathbb{T}'_n/\mathbb{H}(\mathbb{K}_n)) \cong \prod_{g \in \Delta_m} C_{p^n}.$$

Since  $q$  is ramified in  $\mathbb{L}_n/\mathbb{K}_n$ , the residual fields of the ray class subfield  $\mathbb{T}_n := \mathbb{T}_q^-(\mathbb{L}_n)$  are the same as the ones of  $\mathbb{T}'_n$  and thus  $\mathbb{T}_n = \mathbb{H}(\mathbb{L}_n) \cdot \mathbb{T}'_n$ . Let also  $\mathbb{E}'_n \subset \mathbb{T}'_n$  be the maximal subextension with  $p\text{Gal}(\mathbb{E}'_n/\mathbb{H}'_n) = 0$  – thus the  $p$ -elementary extension of  $\mathbb{H}(\mathbb{K}_n)$  contained in  $\mathbb{T}'_n$  – and  $\mathbb{E}_n = \mathbb{E}'_n \cdot \mathbb{H}(\mathbb{L}_n)$ . Since the local extensions  $\mathbb{T}'_{n,\mathfrak{q}'}/\mathbb{H}(\mathbb{K}_n)_{\mathfrak{q}'}$  are cyclotomic, the ramification of  $\mathbb{E}'_n/\mathbb{H}(\mathbb{K}_n)$  is absorbed by  $\mathbb{L}_n/\mathbb{K}_n$  and therefore  $\mathbb{E}_n \subset \mathbb{H}(\mathbb{L}_n)$ .

Consider a class  $x \in A^-(\mathbb{L}_n)$  such that  $0 \neq \beta(x) \in \widehat{H}^1(F, A^-(\mathbb{L}_n))$  and let  $\mathfrak{A} \in x$  be a prime and  $(\alpha) = \mathcal{N}(\mathfrak{A}) \subset \mathbb{K}_n$  be the principal ideal below it. By Lemma 3.7, it follows that  $\pi_N((\alpha/\bar{\alpha})) \neq 0$  and thus the Artin symbol  $y' = \left( \frac{\mathbb{T}'_n/\mathbb{K}_n}{(\alpha/\bar{\alpha})} \right) \in \text{Gal}(\mathbb{T}'_n/\mathbb{K}_n)$  generates a cycle of maximal length in  $\text{Gal}(\mathbb{T}'_n/\mathbb{H}(\mathbb{K}_n))$ , so it acts non trivially in  $\mathbb{E}'_n/\mathbb{H}(\mathbb{K}_n)$ . If  $y \in \text{Gal}(\mathbb{T}_n/\mathbb{L}_n)$  is any lift of  $\varphi(x)$ , i.e.  $y|_{\mathbb{H}(\mathbb{L}_n)} = \varphi(x)$ , then  $\mathcal{N}(y')$  acts non trivially in  $\mathbb{E}'_n/\mathbb{H}(\mathbb{K}_n)$ . The converse holds too. Suppose that  $x \in A^-(\mathbb{L}_n)$  and let  $y \in \text{Gal}(\mathbb{T}_n/\mathbb{L}_n)$  be some lift of  $\varphi(x)$ . If  $\mathcal{N}(y)$  fixes  $\mathbb{H}(\mathbb{K}_n)$  and acts non trivially in  $\mathbb{E}'_n/\mathbb{H}(\mathbb{K}_n)$ , then  $\beta(x) \neq 0$ . Indeed, by choosing  $\mathfrak{A} \in x$  a prime with  $\left( \frac{\mathbb{T}_n/\mathbb{K}_n}{\mathfrak{A}} \right) = y$ , we see that  $I := \mathcal{N}(\mathfrak{A})$  must be a principal ideal, since  $\left( \frac{\mathbb{T}_n/\mathbb{K}_n}{I} \right)$  fixes  $\mathbb{H}(\mathbb{K}_n)$  and moreover, the Artin symbol acts non trivially on  $\mathbb{E}'_n/\mathbb{H}(\mathbb{K}_n)$ , so  $\pi_N(I) \neq 0$ . The claim follows from Lemma 3.7.

Let  $\mathbb{T} = \cup_n \mathbb{T}_n$ . In the projective limit, we conclude that  $x = (x_n)_{n \in \mathbb{N}}$  has  $\beta(x) \neq 0$  iff for any lift  $y \in \text{Gal}(\mathbb{T}/\mathbb{L}_\infty)$  of  $\varphi(x) \in \text{Gal}(\mathbb{H}(\mathbb{L}_\infty))$  the norm  $\mathcal{N}(y)$  fixes  $\mathbb{H}(\mathbb{K}_\infty)$  and acts non trivially in  $\mathbb{E}'/\mathbb{H}(\mathbb{K}_\infty)$ , with  $\mathbb{E} = \cup_n \mathbb{E}'_n$ . Moreover,  $\mathbb{T}/\mathbb{H}(\mathbb{L}_\infty)$  is the product of  $|\Delta_m|$  independent  $\mathbb{Z}_p$ -extensions and



$\mathcal{N}(y)$  is of  $\lambda$ -type. We have thus from 14 a further isomorphism,

$$(16) \quad \hat{H}^1(F, A^-(\mathbb{L}_n)) \cong \text{Gal}(\mathbb{E}'_n/\mathbb{H}(\mathbb{K}_n)) \cong \text{Gal}(\mathbb{E}'_\infty/\mathbb{H}(\mathbb{K}_\infty)).$$

We have thus proved:

**Fact 3.8.** *For every  $n > m$ , a class  $x \in A^-(\mathbb{L}_n)$  has non trivial image  $\beta(x) \in \hat{H}^1(F, A^-(\mathbb{L}_n))$  iff for any lift  $y \in \text{Gal}(\mathbb{T}_n/\mathbb{L}_n)$  of  $\varphi(x) \in \text{Gal}(\mathbb{H}(\mathbb{L}_n)/\mathbb{L}_n)$ , the norm  $y' := \mathcal{N}(y) \in \text{Gal}(\mathbb{T}'_n/\mathbb{K}_n)$  fixes  $\mathbb{H}_n$  and acts non trivially in  $\mathbb{E}'_n$ ; equivalently,  $y'$  generates a maximal cycle in  $\text{Gal}(\mathbb{T}'_n/\mathbb{H}(\mathbb{K}_n))$ . In the projective limit,  $x = (x_n)_{n \in \mathbb{N}}$  has  $\beta(x) \neq 0$  iff for any lift  $y \in \text{Gal}(\mathbb{T}/\mathbb{L}_\infty)$  of  $\varphi(x) \in \text{Gal}(\mathbb{H}(\mathbb{L}_\infty))$  the norm  $\mathcal{N}(y)$  fixes  $\mathbb{H}(\mathbb{K}_\infty)$  and acts non trivially in  $\mathbb{E}'/\mathbb{H}(\mathbb{K}_\infty)$ , with  $\mathbb{E} = \cup_n \mathbb{E}'_n$ . Moreover, there is an exact sequence*

$$(17) \quad 1 \rightarrow \text{Gal}(\mathbb{H}(\mathbb{L}_\infty)/\mathbb{L}_\infty) \rightarrow \text{Gal}(\mathbb{T}/\mathbb{L}_\infty) \rightarrow (\mathbb{Z}_p)^{|\Delta_m|} \rightarrow 1,$$

and thus  $\text{Gal}(\mathbb{T}/\mathbb{K}_\infty)$  is Noetherian  $\Lambda$ -module.

We now prove the Lemma 2.2:

*Proof.* Assume that  $\hat{H}^1(F, (\mathcal{M}^{\text{PA}^-}(\mathbb{L}))) \neq 0$ . Consider the modules  $M_n = \text{Gal}(\mathbb{T}_n/\mathbb{L}_n)$  and  $M = \varprojlim_n M_n = \text{Gal}(\mathbb{T}/\mathbb{L}_\infty)$ . It follows from Fact 3.8 that  $M/\text{Gal}(\mathbb{H}(\mathbb{L}_\infty)) \cong \mathbb{Z}_p^{|\Delta_m|}$ , so  $M$  is a Noetherian  $\Lambda$ -module and the exact

sequence (17) shows that  $M$  is a rigid module and the further premises of Proposition 1.3 hold too, as a consequence of the choice of  $\mathbb{K}$ . Let  $v' \in M$  be such that the restriction  $v = v'|_{\mathbb{H}(\mathbb{L}_\infty)} \in \mathcal{M}({}^p\mathcal{A}^-(\mathbb{L}))$  and it has non trivial image in  $\widehat{H}^1(F, (\mathcal{M}({}^p\mathcal{A}^-(\mathbb{L}))))$ , via the inverse Artin map. In particular  $v \notin \mathcal{L}({}^p\mathcal{A}^-(\mathbb{L}))$  and also  $v' \notin \mathcal{L}(M)$ . Assume that  $\text{ord}(v) \leq p \exp(\mathcal{M}({}^p\mathcal{A}^-(\mathbb{K})))$ . The Proposition 1.3 together with Remark 1.4 imply that  $T^2v' = v'_\mu + v'_\lambda$  is decomposed. But then  $v'_\mu \in \text{Gal}(\mathbb{H}(\mathbb{L}_\infty))$  so it follows from the above Fact that  $\beta(T^2v) = 0$ . Thus, if  $v \in \mathcal{M}({}^p\mathcal{A}^-(\mathbb{L}))$  has order  $\text{ord}(v) \leq p \cdot \exp(\mathcal{M}({}^p\mathcal{A}^-(\mathbb{K})))$ , then  $T^2v \in {}^s\mathcal{M}({}^p\mathcal{A}^-(\mathbb{L}))$ , which completes the proof of Lemma 2.2.  $\square$

**3.5. Proof of the Proposition 1.3.** The proof of the Proposition requires a longer analysis of the growth of modules  $\Lambda x_n$  for indecomposed elements. This will be divided in a sequence of definitions and Lemmata which eventually lead to the proof.

The arguments of this section will take repeatedly advantage of the following elementary Lemma<sup>3</sup>:

**Lemma 3.9.** *Let  $A$  and  $B$  be finitely generated abelian  $p$ -groups denoted additively, and let  $N : B \rightarrow A$ ,  $\iota : A \rightarrow B$  be two  $\mathbb{Z}_p$ -linear maps such that:*

1.  *$N$  is surjective.*
2. *The  $p$ -ranks  $p\text{-rk}(A) = p\text{-rk}(pA) = p\text{-rk}(B) = r$ .*
3.  *$N(\iota(a)) = pa, \forall a \in A$ .*

*Then*

- A. *The inclusion  $\iota(A) \subset pB$  holds unconditionally.*
- B. *We have  $\iota(A) = pB$  and  $B[p] = \text{Ker}(N) \subset \iota(A)$ . Moreover,  $\text{ord}(x) = p \cdot \text{ord}(\iota(N(x)))$  for all  $x \in B$ .*
- C. *If there is a group homomorphism  $T : B \rightarrow B$  with  $\iota(A) \subseteq \text{Ker}(T)$  and  $\nu := \iota \circ N = p + \binom{p}{2}T + O(T^2)$ , then  $\nu = \cdot p$ , i.e.  $\iota(N(x)) = px$  for all  $x \in B$ .*

*Proof.* Since  $A$  and  $B$  have the same  $p$ -rank and  $N$  is surjective, we know that the map  $\overline{N} : B/pB \rightarrow A/pA$  is an isomorphism<sup>4</sup>. Therefore, the map induced by  $N\iota$  on the roof is trivial. Hence  $\bar{\iota} : A/pA \rightarrow B/pB$  is also zero and thus  $\iota(A) \subset pB$ , which confirms the claim A.

The premise  $p\text{-rk}(pA) = p\text{-rk}(A)$  implies that  $p\text{-rk}(A) = p\text{-rk}(\iota(A))$ , as follows from

$$p\text{-rk}(A) \geq p\text{-rk}(\iota(A) = p\text{-rk}(\iota(A)[p]) \geq p\text{-rk}(pA) = p\text{-rk}(A).$$

We now consider the map  $\iota' : A/pA \rightarrow pB/p^2B$  together with  $\overline{N}$ . From the hypotheses we know that  $N\iota'$  is the multiplication by  $p$  isomorphism:  $\cdot p$  :

<sup>3</sup>I owe to Cornelius Greither several elegant ideas which helped simplify my original proof.

<sup>4</sup>For finite abelian  $p$ -groups  $X$  we denote  $R(X) = X/pX$  by *roof* of  $X$  and  $S(X) = X[p]$  is its *socle*

$A/pA \rightarrow pA/p^2A$ , as consequence of  $p\text{-rk}(A) = p\text{-rk}(\iota(A)) = p\text{-rk}(pA)$ . It follows that  $\iota'$  is an isomorphism of  $\mathbb{F}_p$ -vector spaces and hence  $\iota : A \rightarrow pB$  is surjective, so  $\iota(A) = pB$ . Consequently  $|B|/|\iota(A)| = p^r$ . Let  $x \in \text{Ker}(N)$ ; since  $N : B/pB \rightarrow A/pA$  is surjective, the norm does not vanish for  $x \notin pB$ . Consequently, for  $x \in \text{Ker}(N) \subset pB$  we have  $Nx = px = 0$ , so  $\text{Ker}(N) = B[p] \subset \iota(A)$ , so  $\text{Ker}(N) = \iota(A)[p]$ , as claimed.

We now prove that  $\text{ord}(x) = p \cdot \text{ord}(\iota(N(x)))$  for all  $x \in B$ . Consider the following maps  $\pi : B \rightarrow \iota(A), x \mapsto px$  and  $\pi' = \iota \circ N$ . Since  $pB = \iota(A)$ , both maps are surjective and there is an isomorphism  $\phi : pB \rightarrow pB$  such that  $\pi = \phi \circ \pi'$ . Therefore

$$\text{ord}(x)/p = \text{ord}(px) = \text{ord}(\phi(px)) = \text{ord}(\iota(N(x))),$$

and thus  $\text{ord}(x) = p \cdot \text{ord}(\iota(N(x)))$ , as claimed.

For point C. we let  $x \in B$ , so  $px \in pB = \iota(A)$  and thus  $pTx = T(px) = 0$ . Consequently  $Tx \in B[p] \subset \iota(A)$  and thus  $T^2x = 0$ . From the definition of  $\nu = \iota \circ N = p + Tp\frac{p-1}{2} + O(T^2)$  we conclude that  $\nu x = px + \frac{p-1}{2}Tpx + O(T^2)x = px$ , which confirms the claim C. and completes the proof.  $\square$

In this section  $\mathbb{M}_\infty/\mathbb{M}$  is an arbitrary  $\mathbb{Z}_p$ -extension in which all the primes that ramify, ramify completely and  $X' \subseteq Y := \text{Gal}(\mathbb{T}/\mathbb{M}_\infty)$  is a Noetherian  $\Lambda$ -submodule, the limit of the ray class groups  $\text{Gal}(\mathbb{T}_n/\mathbb{M}_n)$  to some ray module associated to the base field with trivial finite part. Thus  $\omega_n x = 0$  implies  $Tx = 0$  by Fact 3.3. The ray class groups  $Y_n = \varphi^{-1}(\text{Gal}(\mathbb{H}(\mathbb{T}_n)/\mathbb{M}_\infty))$  may also coincide with class groups  $\mathbb{A}(\mathbb{M}_n)$ .

We shall apply the Theorem 1.3 to the concrete cases in which  $\mathbb{M} = \mathbb{L}$  and  $Y$  is either  $\text{Gal}(\mathbb{H}^-(\mathbb{L})/\mathbb{L}_\infty)$  or  $\text{Gal}(\mathbb{T}/\mathbb{M}_\infty)$ , where  $\mathbb{T}$  is the injective limit of subfields of ray class groups, defined above.

We denote by  $L, M, D$  the  $\lambda$ -, the  $\mu$ - and the decomposed parts, respectively, of  $X'$ : thus, in the notation of the introduction, we have  $L = \mathcal{L}(X')$ ,  $M = \mathcal{M}(X')$ ,  $D = \mathcal{D}(X')$ . One can for instance think of  $\mathbb{M}$  as  $\mathbb{K}$  in §2 and  $X' = \varphi({}^pA^-(\mathbb{K})) = X^-$  or  $X' = \text{Gal}(\mathbb{T}/\mathbb{H}(\mathbb{K}_\infty))$ .

For  $x \in M$ , the order is naturally defined by  $\text{ord}(x) = \min\{p^k : p^k x = 0\}$ . Since  $X'$  has no finite submodules, it follows that  $\text{ord}(x) = \text{ord}(T^j x)$  for all  $j > 0$ .

We introduce some distances  $d_n : X' \times X' \rightarrow \mathbb{N}$  as follows: let  $x, z \in X'$ ; then

$$d_n(x, z) := p\text{-rk}(\Lambda(x_n - z_n)); \quad d_n(x) = p\text{-rk}(\Lambda x_n).$$

We obviously have  $d_n(x, z) \leq d_n(x, y) + d_n(x, z)$  and  $d_n(x) \geq 0$  with  $d_n(x) = 0$  for the trivial module. Also, if  $f \in \mathbb{Z}_p[T]$  is some distinguished polynomial of degree  $\phi = \deg(f)$ , then  $d_n(x) - \phi \leq d_n(fx) \leq d_n(x)$  for all  $x \in X$ . We shall write  $d(x, y) = \lim_n d_n(x, y)$ . Also, for explicit elements  $u, v \in X'_k$ , we may write  $d(u, v) = p\text{-rk}(\Lambda(u - v))$ . This can be useful for instance when no explicit lifts of  $u, v$  to  $X'$  are known. The simplest fact about the distance is:

**Fact 3.10.** *Let  $x, z \in X'$  be such that  $d_n(x, z) \leq N$  for some fixed bound  $N$  and all  $n > 0$ . Then  $x - z \in L$  and  $N \leq \ell := p\text{-rk}(L)$ . For every fixed  $d \geq p\text{-rk}(L)$  there is an integer  $n_0(d)$  such that for any  $x \in X' \setminus L$  and  $n > n_0$ , if  $d_n(x) \leq d$  then  $x \in \nu_{n, n_0} X'$ .*

*Proof.* The element  $y = x - z$  generates at finite levels modules  $\Lambda y_n$  of bounded rank, so it is neither of  $\mu$ -type nor indecomposed. Thus  $y \in L$  and consequently  $d_n(y) \leq p\text{-rk}(L_n) \leq \ell$  for all  $n$ , which confirms the first claim.

For the second claim, note that if  $x \notin L$ , then  $d_n(x) \rightarrow \infty$ , so the boundedness of  $d_n(x)$  becomes a strong constraint for large  $n$ . Next we recall that  $F(T)x \in M$  and since  $d_n(F(T)x) \leq d_n(x)$ , we may assume that  $x \in M$ . Now  $d_n(x) \leq d$  implies the existence of some distinguished polynomial  $h \in \mathbb{Z}_p[T]$  with  $\deg(h) = d$  and such that  $h(T)x_n = 0$ . The exponent of  $M$  is bounded by  $p^\mu$ , so there is a finite set  $\mathcal{H} \subset \mathbb{Z}_p[T]$  from which  $h$  can take its values. Let now  $n_0$  be chosen such that  $\nu_{n_0, 1} \in (h(T), p^\mu)$  for all  $h \in \mathcal{H}$ . Such a choice is always possible, since  $\nu_{n, 1} = p^\mu \cdot V(T) + T^{p^{n-\mu}-1} \cdot W(T)$  for some  $V(T), W(T) \in \Lambda$ . We may thus choose  $n$  sufficiently large, such that the Euclidean division  $T^{p^{n-\mu}} = q(T)h(T) + r(T)$  yields remainders  $r(T)$  which are divisible by  $p^\mu$  for all  $h \in \mathcal{H}$ . Let  $n_0$  be the smallest such integer. With this choice, for any  $h \in \mathcal{H}$ , it follows that  $h(T)x_n = 0$  implies  $\nu_{n_0, 1}x_n = 0$  and thus  $\nu_{n_0, 1}x = \nu_{n, 1}w$  for some  $w \in X'$ . Then  $\nu_{n_0, 1}(x - \nu_{n, n_0}w) = 0$  and thus  $x = \nu_{n, n_0}w$ , by Fact 3.3 and the assumption on  $X'$ .  $\square$

We pass now to the proof of Proposition 1.3.

*Proof.* Let  $x \in X'$  and suppose that  $l$  is the smallest integer such that  $p^l x \in L$  and let  $f_x(T)$  be the minimal annihilator polynomial of  $p^l x$ , so  $y := f_x(T)x \in M$ , since  $p^l y = 0$ . We claim that

$$p^j x_{n+j} - \iota_{n, n+j}(x_n) \in f_x(T)\Lambda x_{n+j} \subset \Lambda y_{n+j}, \quad \forall j > 0,$$

and in particular  $\iota_{n, n+l}(x_n) = p^l x_{n+l} - h_{n+l}(T)(f_x(T)x_{n+l})$  is decomposed. We let  $n_1 > n_0$  be such that for all  $x = (x_n)_{n \in \mathbb{N}}$  in  $X' \setminus pX'$  we have  $\text{ord}(p^{l+\mu}x_n) > p$ ; here  $n_0$  is the constant established in Fact 3.10 with respect to the bound rank  $d = p\text{-rk}(L) + 1$ . For  $n > n_1$  and  $x \in X' \setminus (D + pX')$ , we have

$$(18) \quad p^l x_{n+l} - \iota_{n, n+l}(x_n) = f_x(T)h_n(T)x_{n+l} \in M_{n+l}.$$

Indeed, consider the modules  $B = \Lambda x_{n+1}/(f_x(T) \cdot \Lambda x_{n+1})$  and  $A = \Lambda x_n/(f_x(T) \cdot \Lambda x_n)$ . Since  $\iota_{n, n+1}x_n \notin f_x(T)\Lambda x_{n+1}$  for  $n > n_1$  – as follows from the condition imposed on the orders – the induced map  $\iota : A \rightarrow B$  is rank preserving. We can thus apply the Lemma 3.9, an  $l$ -fold iteration of which implies the claim (18). We now apply the hypothesis that  $px = c + u \in D$  for some  $c \in L, u \in M[p^{l-1}]$ ; note that the condition (4) allows us to conclude from Lemma 3.9 and  $c \in L$  that  $\iota_{n, n+k}(c_n) = p^k c_{n+k}$  for all  $c = (c_n)_{n \in \mathbb{N}} \in L$ . In

particular,

$$p\omega_n c_{n+1} = \omega_n \iota_{n,n+1}(c_n) = 0, \quad \text{so} \quad \omega_{n+1} c_{n+1} \in L_{n+1}[p] = \iota_{1,n+1}(L_1[p]).$$

as a consequence of the same Lemma. We deduce under the above hypothesis on  $n$ , that

$$p^l x_{n+l} = p^{l-1} c_{n+l} = \iota_{n+1,n+l} c_{n+1} = \iota_{n,n+l}(x_n) + h y_{n+l}.$$

By applying  $\omega_n$  to this identity and using  $\omega_n c_{n+1} \in \iota_{1,n+1}(L_1[p])$ , so  $T\omega_n c_{n+1} = 0$ , we find  $Th(T)\omega_n y_{n+l} = 0$ . The Iwasawa's Theorem VI [5] implies that there is some  $z(n) \in X'$  such that  $Th\omega_n y = \nu_{n+l,1} z(n)$ . Since  $p^l y = 0$ , we have in addition  $p^l \nu_{n+l,1} z(n) = 0$ , so  $p^l z(n) = 0$  and  $z(n) \in M$ , by Fact 3.4. We stress here the dependency of  $z \in X'$  on the choice of  $n > n_1$  by writing  $z(n)$ ; this is however a norm coherent sequence and  $z_m(n)$  will denote its projection in  $X'_m$  for all  $m > 0$ . We obtain  $\nu_{n,1}(T^2 h(T)y - \nu_{n+l,n} z(n)) = 0$ . This implies  $T^2 h(T)y = \nu_{n+l,n} z(n)$ , by the same Fact 3.4. Reinserting this relation in the initial identity, we find

$$(19) \quad \iota_{n,n+l}(T^2 x_n - z_n(n)) = \iota_{n+1,n+l}(T^2 c_{n+1}).$$

We prove that (19) implies that  $T^2 x$  must be decomposed. For this we invoke the Fact 3.10 with respect to the sequence  $w^{(n)} = T^2 x - z(n)$ . We have  $d_n(\iota_{n+1,n+l}(T^2 c_{n+1})) \leq p\text{-rk}(L)$  for all  $n$ ; since  $z(n) \in M$ , and  $T^2 x$  is assumed indecomposed, then  $w^{(n)} \notin D$  either, and the Fact 3.10 together with the choice of  $n_1$  imply that  $w_n^{(n)} \in \nu_{n,n_0} X'$  for all  $n > n_1$ . But then

$$w_n^{(n)} = \iota_{n,n+l}(T^2 x_n - z_n(n)) = \nu_{n,n_0} a_n \in \iota_{n_0,n}(X'_{n_0}).$$

It follows in particular that

$$\text{ord}(T^2 x_n - z_n(n)) \leq p^l \text{ord}(\iota_{n,n+l}(T^2 x_n - z_n(n))) \leq p^l \exp(X'_{n_0}).$$

This holds for arbitrary large  $n$ , and since  $z_n(n) \in M_n$ , we have  $\text{ord}(T^2 x_n - z_n(n)) = \text{ord}(T^2 x_n)$  for  $n > n_1$ . Therefore, the assumption that  $Tx \notin D$  implies that  $\text{ord}(T^2 x_n) \leq p^l \exp(X'_{n_0})$  for all  $n > n_1$ : it is thus uniformly bounded for all  $n$ , which would imply that  $x \in M$ , in contradiction with the assumption  $x \notin D$ . We have thus proved the claim for all  $x \in X' \setminus (D + pX')$ . The general case follows by applying Nakayama to the module  $X'$ .  $\square$

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